

Right coideal subalgebras of quantized universal enveloping algebras of type G_2^*

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Abstract

In this paper we describe the right coideal subalgebras containing all group-like elements of the two-parameter quantum group $U_q(\mathfrak{g})$, where \mathfrak{g} is a simple Lie algebra of type G_2 , while the main parameter of quantization q is not a root of 1. As a consequence, we determine that there are precisely 60 different right coideal subalgebras containing all group-like elements. If the multiplicative order t of q is finite, $t > 4$, $t \neq 6$, then the same classification remains valid for homogeneous right coideal subalgebras of the two-parameter version of the small Lusztig quantum group $u_q(\mathfrak{g})$.

Keywords: Quantum groups, Hopf algebras, Coideal subalgebras, PBW-generators

Mathematics Subject Classification: 16W30, 17B37

1 Introduction

In this paper we continue the classification of the right coideal subalgebras of quantized enveloping algebras started in [7], [8] and [9] by V. K. Kharchenko, A. V. Lara Sagahón and the author. In the second section, following [7] and [9], we introduce the main concepts and general results that are necessary.

In the third section we present the algebra $U_q(\mathfrak{g})$, where \mathfrak{g} is a simple Lie algebra of type G_2 , which has a triangular decomposition

$$U_q(\mathfrak{g}) = U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).$$

Here we also give some previous important results obtained in [9] for the algebra $U_q^+(\mathfrak{g})$. In particular, Theorem 3.1 calculates a PBW-basis for this algebra, and

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Theorem 3.4 completely describes the right coideal subalgebras containing all group-like elements, proving a conjecture of V. K. Kharchenko (see [8]) for the specific case where \mathfrak{g} is a simple Lie algebra of type G_2 . Recently, this conjecture was proved in general by I. Heckenberger and H.-J. Schneider in [1]. In this section we also prove in Lemma 3.5 that the right coideal subalgebras that contain all group-like elements have a triangular decomposition

$$\mathbf{U} = \mathbf{U}^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \mathbf{U}^+$$

and in Lemma 3.6 we present necessary and sufficient conditions to determine when the above tensor product of right coideal subalgebras \mathbf{U}^- and \mathbf{U}^+ of, respectively, $U_q^-(\mathfrak{g})$ and $U_q^+(\mathfrak{g})$ is a right coideal subalgebra of $U_q(\mathfrak{g})$.

In the fourth section we have Theorem 4.1, which is the main result of the paper. Here we prove, using Lemmas 3.5 and 3.6 that we have the following right coideal subalgebras for $U_q(\mathfrak{g})$

	U_{11}	U_{10}	U_9	U_8	U_7	U_1	U_2	U_3	U_4	U_5	U_6	U_{12}
U_{11}^-						✓						
U_{10}^-						✓	✓	✓				
U_9^-						✓	✓	✓	✓			
U_8^-						✓	✓	✓	✓	✓	✓	
U_7^-					✓	✓	✓	✓	✓	✓	✓	✓
U_1^-	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
U_2^-		✓	✓	✓	✓	✓	✓					✓
U_3^-		✓	✓	✓	✓	✓						
U_4^-			✓	✓	✓	✓						
U_5^-				✓	✓	✓						
U_6^-				✓	✓	✓						
U_{12}^-					✓	✓	✓					✓

where $\mathbf{U}^+ = U_i$ and $\mathbf{U}^- = U_j^-$, $1 \leq i, j \leq 12$, are the right coideal subalgebras of $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ described in [9] (see Figure 1 and notations on pages 12-13). In particular, we see that $U_q(\mathfrak{g})$ has precisely 60 different right coideal subalgebras that contain all group-like elements. In the case where q is a root of 1 of order t , $t > 4$, $t \neq 6$, the same results are valid for the homogeneous right coideal subalgebras of $u_q(\mathfrak{g})$.

2 Preliminaries

In this section, following [7] and [9], we recall the basic concepts.

Definition 2.1. Let S be an algebra over a field \mathbf{k} and A its subalgebra with a fixed basis $\{a_j | j \in J\}$. A linearly ordered subset $W \subseteq S$ is said to be a set of *PBW-generators of S over A* if there exists a function $h : W \rightarrow \mathbb{Z}^+ \cup \infty$, called

the height function, such that the set of all products

$$a_j w_1^{n_1} w_2^{n_2} \dots w_k^{n_k}, \quad (1)$$

where $j \in J$, $w_1 < w_2 < \dots < w_k \in W$, $n_i < h(w_i)$, $1 \leq i \leq k$ is a basis of S . The value $h(w)$ is referred to as the *height* of w in W . If $A = \mathbf{k}$ is the ground field, then we shall call W simply as a set of PBW-generators of S .

Definition 2.2. Let W be a set of PBW-generators of S over a subalgebra A . Suppose that the set of all words in W as a free monoid has its own order \prec (that is, $a \prec b$ implies $cad \prec cbd$ for all words $a, b, c, d \in W$). A *leading word* of $s \in S$ is the maximal word $m = w_1^{n_1} w_2^{n_2} \dots w_k^{n_k}$ that appears in the decomposition of s in the basis (1). A *leading term* of s is the sum am of all terms $\alpha_i a_i m$ that appear in the decomposition of s in the basis (1), where m is the leading word of s .

Definition 2.3. A Hopf algebra H is said a *character Hopf algebra* if the group G of all group-like elements is commutative and H is generated over $\mathbf{k}[G]$ by skew primitive semi-invariants $a_i, i \in I$:

$$\Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i, \quad g^{-1} a_i g = \chi^i(g) a_i, \quad g, g_i \in G,$$

where $\chi^i, i \in I$, are characters of the group G .

Let us associate a quantum variable x_i to a_i . For each word u in $X = \{x_i | i \in I\}$ we denote by g_u an element of G that appears from u by replacing each x_i with g_i . In the same way we denote by χ^u a character that appears from u by replacing each x_i with χ^i . We define a bilinear skew commutator on homogeneous linear combinations of words by the formula

$$[u, v] = uv - \chi^u(g_v)vu, \quad (2)$$

where we use the notation $\chi^u(g_v) = p_{uv} = p(u, v)$.

The group G acts on the free algebra $\mathbf{k}\langle X \rangle$ by $g^{-1}ug = \chi(g)u$, where u is an arbitrary monomial in X . The skew group algebra $G\langle X \rangle$ has the natural Hopf algebra structure

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad i \in I, \quad \Delta(g) = g \otimes g.$$

We fix a Hopf algebra homomorphism

$$\xi : G\langle X \rangle \rightarrow H, \quad \xi(x_i) = a_i, \quad \xi(g) = g, \quad i \in I, \quad g \in G.$$

Definition 2.4. A *constitution* of a word u in $G \cup X$ is a family of non-negative integers $\{m_x, x \in X\}$ such that u has m_x occurrences of x . Certainly almost all m_x in the constitution are zero.

Let us fix an arbitrary complete order $<$ on the set X , and let Γ^+ be the free additive (commutative) monoid generated by X . The monoid Γ^+ is a completely ordered monoid with respect to the following order:

$$m_1 x_{i_1} + m_2 x_{i_2} + \dots + m_k x_{i_k} > m'_1 x_{i_1} + m'_2 x_{i_2} + \dots + m'_k x_{i_k} \quad (3)$$

if the first from the left nonzero number in $(m_1 - m'_1, m_2 - m'_2, \dots, m_k - m'_k)$ is positive, where $x_{i_1} > x_{i_2} > \dots > x_{i_k}$ in X . We associate a formal degree $D(u) = \sum_{x \in X} m_x x \in \Gamma^+$ to a word u in $G \cup X$, where $\{m_x | x \in X\}$ is the constitution of u . Respectively, if $f = \sum \alpha_i u_i \in G\langle X \rangle$, $0 \neq \alpha_i \in \mathbf{k}$ then

$$D(f) = \max_i \{D(u_i)\}. \quad (4)$$

On the set of all words in X we fix the lexicographical order with the priority from the left to the right, where a proper beginning of a word is considered to be greater than the word itself.

According to [6, Theorem 1.1], every right coideal subalgebra \mathbf{U} that contains all group-like elements has a PBW-basis T over $\mathbf{k}[G]$ which can be extended up to a PBW-basis P_T of H . The set T of PBW-generators for \mathbf{U} can be obtained from the PBW-basis of H given in [4, Theorem 2] in the following way.

Suppose that for a given PBW-generator $[u]$ there exists an element $c \in \mathbf{U}$

$$c = [u]^s + \sum \alpha_i W_i + \sum \beta_j V_j \quad (5)$$

where W_i are the basis words starting with a PBW-generator $[v] < [u]$, $D(W_i) = sD(u)$, and V_j are basis words multiplied from the left by a group-like element with $D(V_j) < sD(u)$. We fix one of the elements with minimal s and denote it by c_u . Thus, for every PBW-generator $[u]$ we have at most one element c_u . We define the height function by the following lemma.

Lemma 2.5. [6, Lemma 4.3] *In the representation (5) of the chosen element c_u , either $s = 1$ or $p(u, u)$ is a primitive t -th root of 1 and $s = t$, or (in the case of positive characteristic) $s = t(\text{char } \mathbf{k})^r$.*

If the height of $[u]$ in H is infinite, then the height of c_u in \mathbf{U} is defined to be infinite as well. If the height of $[u]$ in H equals t and $p(u, u)$ is a primitive t -th root of 1, then, due to the above lemma, $s = 1$ (note that in the representation (5) the number s is less than the height of $[u]$). In this case, the height of c_u in \mathbf{U} is supposed to be t as well. If the characteristic l is positive and the height of $[u]$ in H equals tl^r , then we define the height of c_u in \mathbf{U} to be equal to tl^r/s (thus, in characteristic zero the height of c_u in \mathbf{U} always equals the height of $[u]$ in H).

Proposition 2.6. [6, Proposition 4.4] *An element $c \in H$ belongs to \mathbf{U} if and only if all PBW-generators in the PBW-decomposition of c with respect to P_T belong to T . In particular, the set T of all chosen c_u with the above defined height function is a set of PBW-generators of \mathbf{U} over $\mathbf{k}[G]$.*

Definition 2.7. Let $C = \| a_{ij} \|$ be a generalized Cartan matrix symmetrizable by $D = \text{diag}(d_1, \dots, d_n)$, $d_i a_{ij} = d_j a_{ji}$. Denote by \mathfrak{g} a Kac-Moody algebra defined by C (see [2]). Suppose that the quantification parameters $p_{ij} = p(x_i, x_j) = \chi^i(g_j)$ are related by

$$p_{ii} = q^{d_i}, \quad p_{ij} p_{ji} = q^{d_i a_{ij}}, \quad 1 \leq i, j \leq n. \quad (6)$$

The *multiparameter quantization* $U_q^+(\mathfrak{g})$ of the Borel subalgebra \mathfrak{g}^+ is a character Hopf algebra generated by $x_1, \dots, x_n, g_1, \dots, g_n$ and defined by Serre relations with the skew brackets (2) in place of the Lie operation:

$$[[\dots [[x_i, x_j], x_j], \dots], x_j] = 0, \quad 1 \leq i \neq j \leq n, \quad (7)$$

where x_j appears $1 - a_{ji}$ times.

We note that relations (6) diminish the number of free parameters from n^2 to $\frac{n(n-1)}{2} + 1$. In particular, if $n = 2$ we have just a two-parameter family of quantizations.

By [3, Theorem 6.1], the left sides of the relations (7) are skew primitive elements in $G\langle X \rangle$. Therefore, the ideal generated by these elements is a Hopf ideal, while $U_q^+(\mathfrak{g})$ indeed has a natural character Hopf algebra structure.

Definition 2.8. If the multiplicative order t of q is finite, then we define $u_q^+(\mathfrak{g})$ as $G\langle X \rangle / \Lambda$, where Λ is the biggest Hopf ideal in $G\langle X \rangle^{(2)}$, which is the set (an ideal) of noncommutative polynomials without free and linear terms. From [5, Lemma 2.2], this is a Γ^+ -homogeneous ideal. Certainly Λ contains all skew primitive elements of $G\langle X \rangle^{(2)}$ (each one of them generates a Hopf ideal). Hence, by [3, Theorem 6.1], relations (7) are still valid in $u_q^+(\mathfrak{g})$.

Definition 2.9. Consider a new set of variables $X^- = \{x_1^-, x_2^-, \dots, x_n^-\}$. Suppose that an Abelian group F , generated by the elements f_1, f_2, \dots, f_n , acts on the linear space spanned by X^- so that $(x_i^-)^{f_j} = p_{ji}^{-1} x_i^-$, where p_{ij} are the same parameters (6) that define $U_q^+(\mathfrak{g})$. The relations (6) are invariant under substitutions $p_{ij} \leftarrow p_{ji}^{-1}$, $q \leftarrow q^{-1}$. This allows us to define the character Hopf algebra $U_q^-(\mathfrak{g})$ as $U_{q^{-1}}^+(\mathfrak{g})$ with the characters χ_-^i , $1 \leq i \leq n$ such that $\chi_-^i(f_j) = p_{ji}^{-1}$.

We may extend the characters χ^i on $G \times F$ in the following way

$$\chi^i(f_j) := p_{ji} = \chi^j(g_i).$$

Indeed, if $\prod_k f_k^{m_k} = 1$ in F , then the application to x_i^- implies $\prod_k p_{ki}^{-m_k} = 1$, hence $\chi^i(\prod_k f_k^{m_k}) = \prod_k p_{ki}^{-m_k}$ equals 1 as well. In the same way, we may extend the characters χ_-^i on $G \times F$ so that

$$\chi_-^i = (\chi^i)^{-1}$$

In what follows we denote by H the quotient group $(G \times F)/N$, where N is an arbitrary subgroup with $\chi^i(N) = 1$, $1 \leq i \leq n$. For example, if the quantification parameters satisfy additional symmetry conditions $p_{ij} = p_{ji}$, $1 \leq i, j \leq n$, as in the case for the original Drinfeld-Jimbo and Lusztig quantifications, then $\chi^i(g_k^{-1} f_k) = p_{ik}^{-1} p_{ki} = 1$, and we may take N to be the subgroup generated by $g_k^{-1} f_k$, $1 \leq k \leq n$. In this particular case the groups H , G , F may be identified.

In the general case, without loss of generality we may suppose that $G, F \subseteq H$. Certainly χ^i , $1 \leq i \leq n$ are characters of H and H still acts on the space spanned by $X \cup X^-$ by means of these characters and their inverses.

Definition 2.10. Consider the skew group algebra $H\langle X \cup X^- \rangle$ as a character Hopf algebra:

$$\begin{aligned}\Delta(x_i) &= x_i \otimes 1 + g_i \otimes x_i, & \Delta(x_i^-) &= x_i^- \otimes 1 + f_i \otimes x_i^-, \\ g^{-1}x_i g &= \chi^i(g) \cdot x_i, & g^{-1}x_i^- g &= (\chi^i)^{-1}(g) \cdot x_i^-, & g \in G, \\ f^{-1}x_i f &= \chi_-^i(f) \cdot x_i, & f^{-1}x_i^- f &= (\chi_-^i)^{-1}(f) \cdot x_i^-, & f \in F.\end{aligned}\tag{8}$$

We define the algebra $U_q(\mathfrak{g})$ as a quotient of $H\langle X \cup X^- \rangle$ by the following relations:

$$\begin{aligned}[[\dots [x_i, x_j], x_j], \dots], x_j] &= 0, & 1 \leq i \neq j \leq n, \\ [[\dots [x_i^-, x_j^-], x_j^-], \dots], x_j^-] &= 0, & 1 \leq i \neq j \leq n,\end{aligned}$$

where x_j and x_j^- appear $1 - a_{ji}$ times and

$$[x_i, x_j^-] = \delta_i^j (1 - g_i f_i), \quad 1 \leq i, j \leq n \tag{10}$$

where the brackets are defined on $H\langle X \cup X^- \rangle$ by the structure of character Hopf algebra as in (2). Since due to (6) and [3, Theorem 6.1], all polynomials in the above relations are skew primitive in $H\langle X \cup X^- \rangle$, they define a Hopf ideal of $H\langle X \cup X^- \rangle$, that is, the natural homomorphism

$$H\langle X \cup X^- \rangle \rightarrow U_q(\mathfrak{g})$$

defines a Hopf algebra structure on $U_q(\mathfrak{g})$.

Definition 2.11. If q has finite multiplicative order, then $u_q(\mathfrak{g})$ is defined by relations (10) and $u = 0, u \in \Lambda, u^- = 0, u^- \in \Lambda^-$, where Λ, Λ^- are the biggest Hopf ideals in $G\langle X \rangle^{(2)}$ and $F\langle X^- \rangle^{(2)}$, which are sets (ideals) of noncommutative polynomials without free and linear terms.

Both algebras $U_q(\mathfrak{g})$ and $u_q(\mathfrak{g})$ are graded by Γ , the additive group generated by Γ^+ , provided that we put $D(x_i^-) = -D(x_i) = -x_i, D(H) = 0$ since in this way the relations (10) become homogeneous.

Corollary 2.12. [7, Corollary 3.3] *If q is not a root of 1 and the Cartan matrix $C = \|a_{ij}\|$ is of finite type, then every subalgebra \mathbf{U} of $U_q(\mathfrak{g})$ containing H is Γ -homogeneous.*

This Corollary is not true for $u_q(\mathfrak{g})$ if q is a root of 1. In this case, we may have subalgebras that contain H and that are not homogeneous.

From [7], the subalgebra of $U_q(\mathfrak{g})$ generated by G and x_1, x_2, \dots, x_n is isomorphic to $U_q^+(\mathfrak{g})$. Analogously, the subalgebra of $U_q(\mathfrak{g})$ generated by F and $x_1^-, x_2^-, \dots, x_n^-$ is isomorphic to $U_q^-(\mathfrak{g})$. Moreover, we have the following triangular decomposition:

$$U_q(\mathfrak{g}) = U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}). \tag{11}$$

We also have the same fact for $u_q(\mathfrak{g})$:

$$u_q(\mathfrak{g}) = u_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} u_q^+(\mathfrak{g}). \tag{12}$$

Now we have all the necessary definitions and results for the following sections.

3 The algebras $U_q(\mathfrak{g})$, $u_q(\mathfrak{g})$ when \mathfrak{g} is the simple Lie algebra of type G_2

Now we are going to consider the algebras $U_q(\mathfrak{g})$, when q is not a root of 1, and $u_q(\mathfrak{g})$, when q has finite multiplicative order t , $t > 4$, $t \neq 6$, where \mathfrak{g} is the simple Lie algebra of type G_2 . Let us remember that the algebra $U_q(\mathfrak{g})$ is defined by generators $x_1, x_2, x_1^-, x_2^-, g_1, g_2, f_1, f_2$ and relations

$$\begin{aligned} [[[[x_1, x_2], x_2], x_2], x_2] &= 0, & [x_1, [x_1, x_2]] &= 0, \\ [[[[x_1^-, x_2^-], x_2^-], x_2^-], x_2^-] &= 0, & [x_1^-, [x_1^-, x_2^-]] &= 0, \\ [x_i, x_j^-] &= \delta_i^j (1 - g_i f_i), & i, j &= 1, 2, \end{aligned} \quad (13)$$

according to Definition 2.10, where $p_{11} = q^3$, $p_{22} = q$ and $p_{12}p_{21} = q^{-3}$. In this way we have a two-parameter family of quantizations. We also have that the generators g_1, g_2, f_1, f_2 are all invertible, commute with each other and

$$x_i g_j = p_{ij} g_j x_i, \quad x_i f_j = p_{ji} f_j x_i, \quad x_i^- g_j = p_{ij}^{-1} g_j x_i^-, \quad x_i^- f_j = p_{ji}^{-1} f_j x_i^-.$$

The algebra $u_q(\mathfrak{g})$ is defined as in Definition 2.11 and has the same relations (13).

As we have seen in the preliminaries, the algebras $U_q(\mathfrak{g})$ and $u_q(\mathfrak{g})$ have the triangular decompositions (11) and (12). Now we will give some results that are proved for $U_q^+(\mathfrak{g})$ and $u_q^+(\mathfrak{g})$.

Theorem 3.1. [9, Theorem 3.4] *If q is not a root of 1, then the following elements*

$$\begin{aligned} [A] &= x_1, \\ [B] &= [x_1, x_2], \\ [C] &= [[x_1, x_2], [x_1, x_2], x_2], \\ [D] &= [[x_1, x_2], x_2], \\ [E] &= [[[x_1, x_2], x_2], x_2], \\ [F] &= x_2. \end{aligned} \quad (14)$$

form a set of PBW-generators for $U_q^+(\mathfrak{g})$ over $\mathbf{k}[G]$, and each PBW-generator has infinite height. If we suppose that $x_1 > x_2$, then $A > B > C > D > E > F$.

Theorem 3.2. [9, Theorem 3.6] *If q has finite multiplicative order t , $t > 4$, $t \neq 6$, then the elements from list (14) form a set of PBW-generators for $u_q^+(\mathfrak{g})$ over $\mathbf{k}[G]$. The height h of $[u] \in \{[B], [D], [F]\}$ equals t . For $[u] \in \{[A], [C], [E]\}$ we have $h = t$ if 3 is not a divisor of t and $h = \frac{t}{3}$ otherwise. In all cases $[u]^h = 0$ in $u_q^+(\mathfrak{g})$.*

Lemma 3.3. [9, Corollary 3.7] *The exponent s given in (5) is 1 for every $[u]$ in the PBW-basis.*

Theorem 3.4. [9, Theorem 1.1] *If q is not a root of 1, the lattice of right coideal subalgebras containing $\mathbf{k}[G]$ of $U_q^+(\mathfrak{g})$ is given in the Figure 1. If q has multiplicative order $t > 4$, $t \neq 6$, the same figure is the lattice of homogeneous right coideal subalgebras containing $\mathbf{k}[G]$ of $u_q^+(\mathfrak{g})$.*

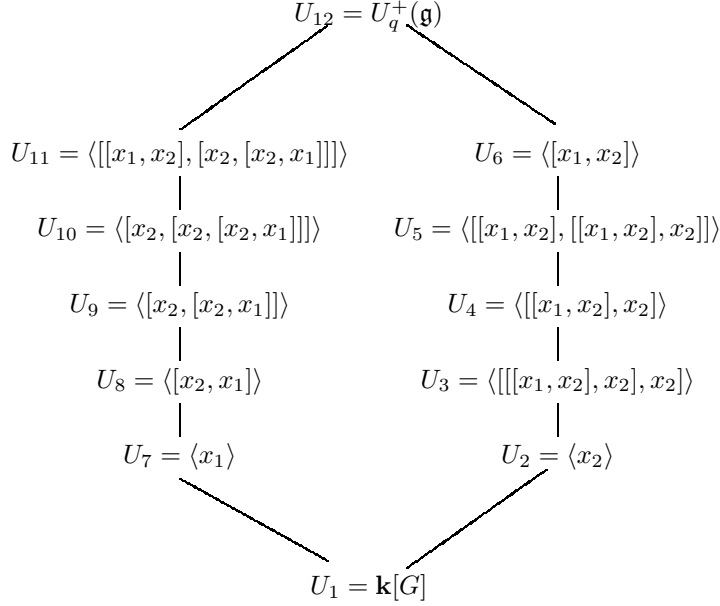


Figure 1: Lattice of Right Coideal Subalgebras

Since the algebra $U_q^-(\mathfrak{g})$ is defined as $U_{q^{-1}}^+(\mathfrak{g})$, a similar result is valid for the negative quantum Borel algebra too.

At this point we would like to say that any (Γ -homogeneous) right coideal subalgebra of $U_q(\mathfrak{g})$ (respectively, $u_q(\mathfrak{g})$) that contains $\mathbf{k}[H]$ has the triangular decomposition, and that for any two right coideal subalgebras $\mathbf{k}[F] \subseteq \mathbf{U}^- \subseteq U_q^-(\mathfrak{g})$, $\mathbf{k}[G] \subseteq \mathbf{U}^+ \subseteq U_q^+(\mathfrak{g})$ (respectively, $\mathbf{k}[F] \subseteq \mathbf{U}^- \subseteq u_q^-(\mathfrak{g})$, $\mathbf{k}[G] \subseteq \mathbf{U}^+ \subseteq u_q^+(\mathfrak{g})$) the tensor product

$$\mathbf{U} = \mathbf{U}^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \mathbf{U}^+ \quad (15)$$

is a right coideal subalgebra. However this is not true: the tensor product (15) is not always a subalgebra. We have instead the next two lemmas.

Lemma 3.5. *If q is not a root of 1 then every right coideal subalgebra $\mathbf{U} \supseteq \mathbf{k}[H]$ of $U_q(\mathfrak{g})$ has a decomposition (15), where $\mathbf{U}^+ \supseteq \mathbf{k}[G]$ and $\mathbf{U}^- \supseteq \mathbf{k}[F]$ are right coideal subalgebras of $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$, respectively. If q has finite multiplicative order t , $t > 4$, $t \neq 6$, then this is the case for Γ -homogeneous right coideal subalgebras of $u_q(\mathfrak{g})$.*

Proof. Due to the triangular decompositions (11) and (12), the set $P = P_- \cup P_+$ is a set of PBW-generators for $U_q(\mathfrak{g})$ (respectively, $u_q(\mathfrak{g})$) over $\mathbf{k}[H]$, where P_- is the set of PBW-generators for $U_q^-(\mathfrak{g})$

$$\{x_1^-, x_2^-, [x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-], [[[x_1^-, x_2^-], x_2^-], x_2^-], [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]]\}$$

and P_+ is the set of PBW-generators for $U_q^+(\mathfrak{g})$

$$\{x_1, x_2, [x_1, x_2], [[x_1, x_2], x_2], [[[x_1, x_2], x_2], x_2], [[x_1, x_2], [[x_1, x_2], x_2]]\}.$$

Let us fix the following order on the skew primitive generators

$$x_1 > x_2 > x_1^- > x_2^-.$$

By Proposition 2.6 and Lemma 3.3, the subalgebra \mathbf{U} has PBW-generators of the form

$$[u] + \sum \alpha_i W_i + \sum \beta_j V_j \in \mathbf{U},$$

where $[u] \in P$, W_i are the basis words starting with PBW-generators smaller than $[u]$, $D(W_i) = D(u)$, and V_j are basis words multiplied from the left by a group-like element with $D(V_j) < D(u)$. From the definition of the degree given in (4), all W_i have the same constitution as the leading term $[u]$. Thus all W_i 's and the leading term $[u]$ belong to the same component of the triangular decomposition (that is, they have only elements from X , or only elements from X^- , they can't have both). It remains to show that there are no terms V_j .

If q is not a root of 1, by Corollary 2.12, the algebra \mathbf{U} is Γ -homogeneous. Hence (in both cases) the PBW-generators may be chosen to be Γ -homogeneous as well. It means that $a - c = a_j - c_j$ and $b - d = b_j - d_j$ supposing that $ax_1 + bx_2 + cx_1^- + dx_2^-$ is the degree of $[u]$ and $a_jx_1 + b_jx_2 + c_jx_1^- + d_jx_2^-$ is the degree of V_j , for every j . However it contradicts the hypothesis that $D(V_j) < D(u)$, as it will be shown.

If the leading term $[u] \in P_-$, then $a = b = 0$, what provides $a_j = b_j = 0$, from the order defined in (3). So $c = c_j$ and $d = d_j$, and $D(V_j) = D(u)$. If $[u] \in P_+$, then $c = d = 0$ and $a = a_j - c_j$, $b = b_j - d_j$. But then $a_j \geq a$, $b_j \geq b$, $c_j \geq c$ and $d_j \geq d$, what gives us $D(V_j) \geq D(u)$.

Now we can see that all PBW-generators belong either to $U_q^-(\mathfrak{g})$ (respectively, $u_q^-(\mathfrak{g})$) or $U_q^+(\mathfrak{g})$ (respectively, $u_q^+(\mathfrak{g})$). Therefore, \mathbf{U} has the decomposition (15). \square

Lemma 3.6. [7, Lemma 9.3] *The tensor product (15) is a right coideal subalgebra if and only if*

$$[\mathbf{U}^+, \mathbf{U}^-] \subseteq \mathbf{U}^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \mathbf{U}^+. \quad (16)$$

In fact, it was also proved by Kharchenko that, to verify the inclusion (16), we just need to verify that

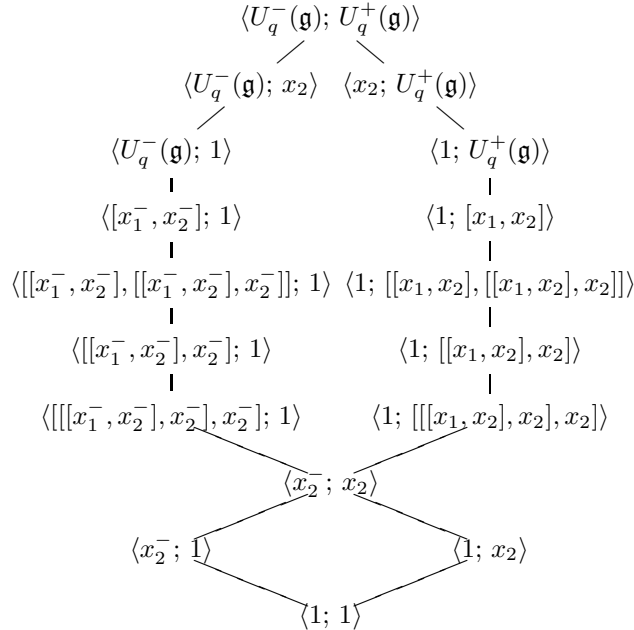
$$[[u^+], [u^-]] \in \mathbf{U}^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \mathbf{U}^+.$$

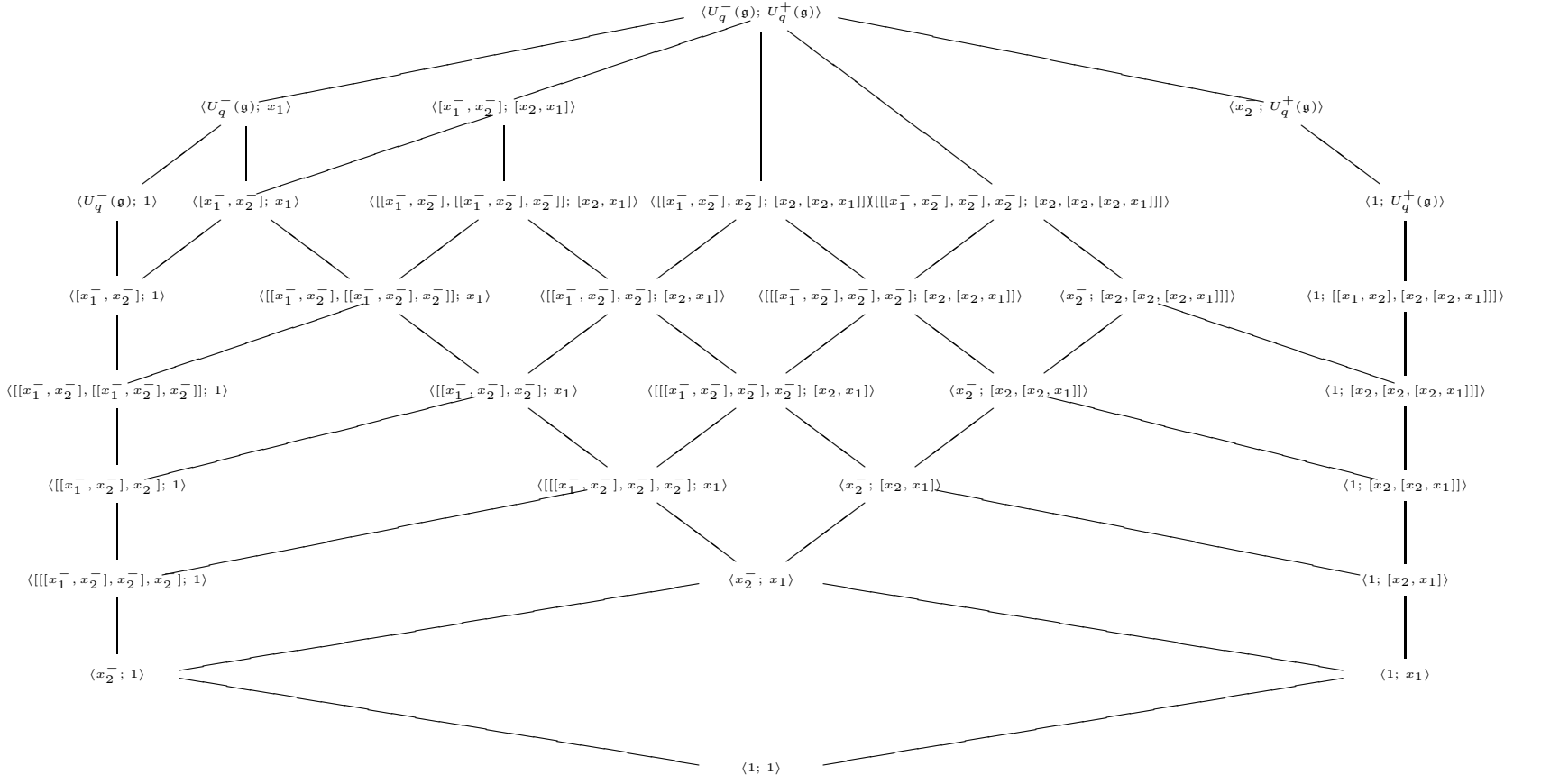
for every PBW-generators $[u^+]$ and $[u^-]$ of \mathbf{U}^+ and \mathbf{U}^- , respectively.

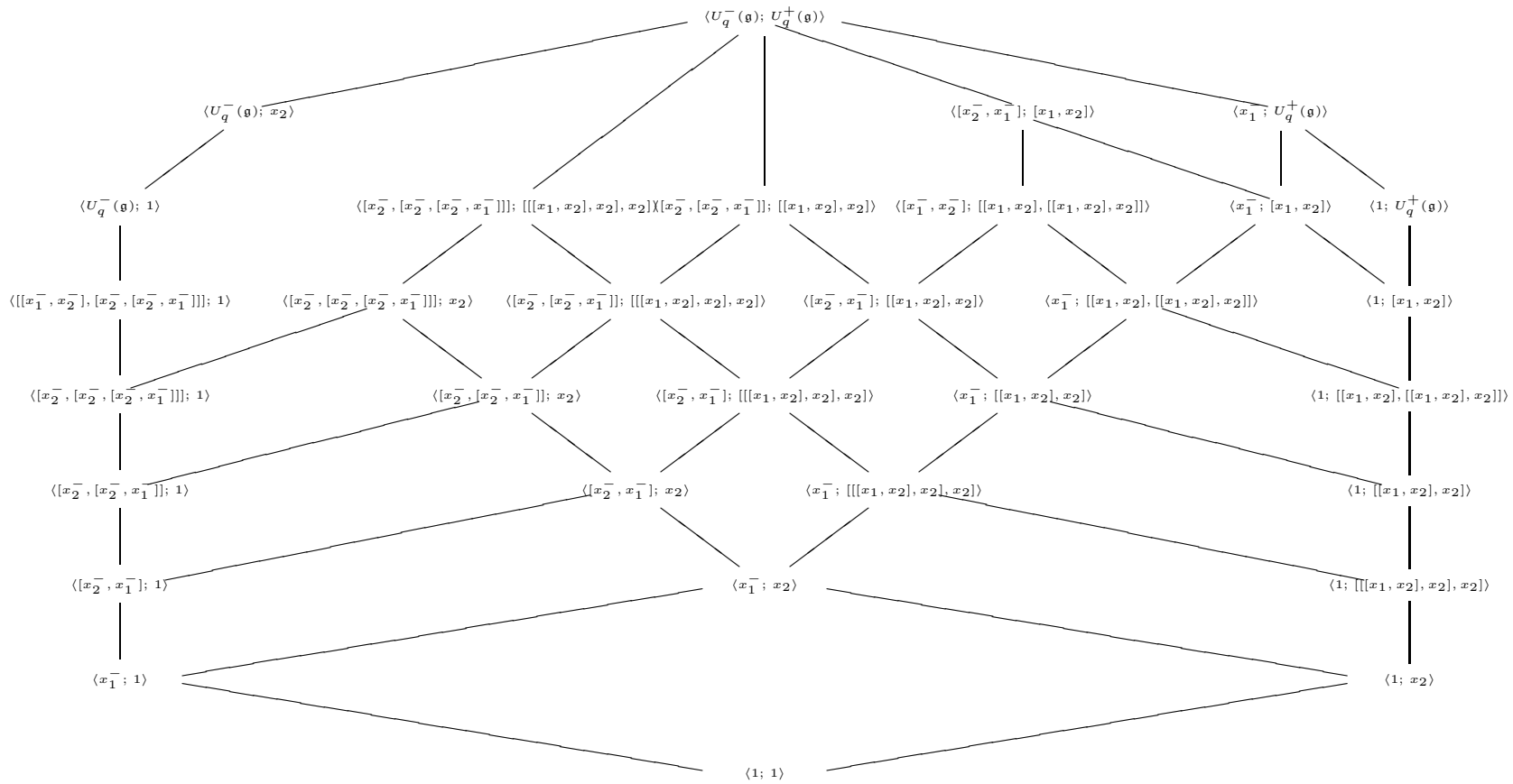
4 Right Coideal Subalgebras of $U_q(\mathfrak{g})$, $u_q(\mathfrak{g})$

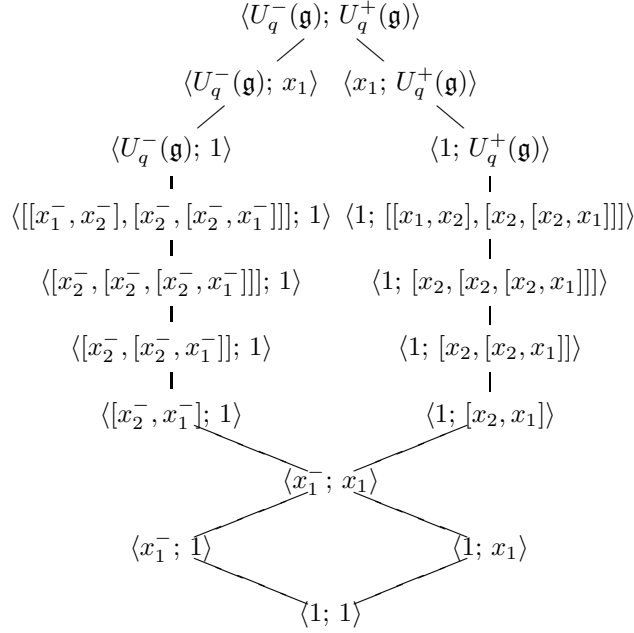
The idea for this section is, using Theorem 3.4 and Lemma 3.6, to describe the lattice of (homogeneous) right coideal subalgebras containing $\mathbf{k}[H]$ of $U_q(\mathfrak{g})$ (respectively, $u_q(\mathfrak{g})$). This lattice is represented by four different figures. The first one is the combination of the right coideal subalgebras from the right side of Figure 1 with the right side of the lattice of right coideal subalgebras of $U_q^-(\mathfrak{g})$. The second is the combination of the left sides of both lattices. The last two analyze the left side with the right side, and the right side with the left side. In what follows we denote by $\langle u^-; v \rangle$ the space $\langle u^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle v \rangle$, where u, v are generators of the right coideal subalgebras of quantum Borel subalgebras from Figure 1. If we get $\langle u_1^-; v_1 \rangle$, $\langle u_2^-; v_2 \rangle$ two right coideal subalgebras from two different figures, then their supremum is the smallest right coideal subalgebra containing $\langle u_3^-; v_3 \rangle$ and their infimum is the greatest right coideal subalgebra contained in $\langle u_0^-; v_0 \rangle$, where $\langle u_3^- \rangle$ is the supremum of $\langle u_1^- \rangle$ and $\langle u_2^- \rangle$, $\langle v_3 \rangle$ is the supremum of $\langle v_1 \rangle$ and $\langle v_2 \rangle$, $\langle u_0^- \rangle$ is the infimum of $\langle u_1^- \rangle$ and $\langle u_2^- \rangle$ and $\langle v_0 \rangle$ is the infimum of $\langle v_1 \rangle$ and $\langle v_2 \rangle$. Our main result says that the space $\langle u^-; v \rangle$ is a right coideal subalgebra if and only if it appears on one of the following four figures.

Theorem 4.1. *If $q \in \mathbf{k}$ is not a root of 1, the lattice of right coideal subalgebras containing $\mathbf{k}[H]$ of $U_q(\mathfrak{g})$ is given by the following four figures. If q has finite multiplicative order t , $t > 4, t \neq 6$, the same lattice represents the homogeneous right coideal subalgebras containing $\mathbf{k}[H]$ of $u_q(\mathfrak{g})$.*









Proof. From Lemma 3.5, all (homogeneous) right coideal subalgebras containing $\mathbf{k}[H]$ of $U_q(\mathfrak{g})$ (respectively, $u_q(\mathfrak{g})$) have the form

$$\mathbf{U} = \mathbf{U}^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \mathbf{U}^+, \quad (17)$$

where $\mathbf{U}^+, \mathbf{U}^-$ are (homogeneous) right coideal subalgebras containing $\mathbf{k}[G], \mathbf{k}[F]$ of $U_q^+(\mathfrak{g}), U_q^-(\mathfrak{g})$ (respectively, $u_q^+(\mathfrak{g}), u_q^-(\mathfrak{g})$). Now we just have to see which combinations are right coideal subalgebras and which ones are not.

First let us remember that, from Theorem 3.4 the (homogeneous) right coideal subalgebras of $U_q^+(\mathfrak{g})$ (respectively, $u_q^+(\mathfrak{g})$) with respective PBW-generators over $\mathbf{k}[G]$ are:

- $U_1 = \mathbf{k}[G]$
- $U_2 = \langle x_2 \rangle$
 $\{x_2\}$
- $U_3 = \langle [[[x_1, x_2], x_2], x_2] \rangle$
 $\{x_2; [[[x_1, x_2], x_2], x_2]\}$
- $U_4 = \langle [[x_1, x_2], x_2] \rangle$
 $\{x_2; [[[x_1, x_2], x_2], x_2]; [[x_1, x_2], x_2]\}$
- $U_5 = \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle$
 $\{x_2; [[[x_1, x_2], x_2], x_2]; [[x_1, x_2], x_2]; [[x_1, x_2], [[x_1, x_2], x_2]]\}$

- $U_6 = \langle [x_1, x_2] \rangle$
 $\{x_2; [[[x_1, x_2], x_2], x_2]; [[x_1, x_2], x_2]; [[x_1, x_2], [[x_1, x_2], x_2]]; [x_1, x_2]\}$
- $U_7 = \langle x_1 \rangle$
 $\{x_1\}$
- $U_8 = \langle [x_2, x_1] \rangle$
 $\{x_1; [x_2, x_1]\}$
- $U_9 = \langle [x_2, [x_2, x_1]] \rangle$
 $\{x_1; [x_2, x_1]; [x_2, [x_2, x_1]]\}$
- $U_{10} = \langle [x_2, [x_2, [x_2, x_1]]] \rangle$
 $\{x_1; [x_2, x_1]; [x_2, [x_2, x_1]]; [x_2, [x_2, [x_2, x_1]]]\}$
- $U_{11} = \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle$
 $\{x_1; [x_2, x_1]; [x_2, [x_2, x_1]]; [x_2, [x_2, [x_2, x_1]]]; [[x_1, x_2], [x_2, [x_2, x_1]]]\}$
- $U_{12} = U_q^+(\mathfrak{g})$
 $\{x_2; [[[x_1, x_2], x_2], x_2]; [[x_1, x_2], x_2]; [[x_1, x_2], [[x_1, x_2], x_2]]; [x_1, x_2]; x_1\}$

Since $U_q^-(\mathfrak{g}) = U_{q^{-1}}^+(\mathfrak{g})$, the right coideal subalgebras containing $\mathbf{k}[F]$ of $U_q^-(\mathfrak{g})$ are exactly the same, just substituting x_1 for x_1^- and x_2 for x_2^- . From Lemmas 3.5 and 3.6, in order to find out the (homogeneous) right coideal subalgebras containing $\mathbf{k}[H]$ of $U_q(\mathfrak{g})$ (respectively, $u_q(\mathfrak{g})$) we just have to calculate the skew-commutators between the possible PBW-generators, which are just listed here. Since we do not need to know the exact coefficients, for simplification of the expressions we define the following relation

$$u \sim v \quad \text{if and only if} \quad u = \alpha v,$$

where $\alpha \in \mathbf{k}, \alpha \neq 0$. For the coefficients $\alpha_i, i = 1, \dots, 8$, that appear on some expressions, we also have $\alpha_i \in \mathbf{k}$ and $\alpha_i \neq 0$, except for α_1 which may be zero. All calculations will be shown at the appendix, where these coefficients are also specified.

$$\textbf{(A1)} \quad [x_1, x_1^-] \sim 1 - g_1 f_1,$$

$$\textbf{(A2)} \quad [x_1, x_2^-] \sim 0,$$

$$\textbf{(A3)} \quad [x_1, [x_1^-, x_2^-]] \sim x_2^- g_1 f_1,$$

$$\textbf{(A4)} \quad [x_1, [x_2^-, x_1^-]] \sim x_2^-,$$

$$\textbf{(A5)} \quad [x_1, [[x_1^-, x_2^-], x_2^-]] \sim (x_2^-)^2 g_1 f_1,$$

$$\textbf{(A6)} \quad [x_1, [[[x_1^-, x_2^-], x_2^-], x_2^-]] \sim (x_2^-)^3 g_1 f_1,$$

- (A7) $[x_1, [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]]] \sim x_2^- [[x_1^-, x_2^-], x_2^-] g_1 f_1 + \alpha_1 [[[x_1^-, x_2^-], x_2^-], x_2^-] g_1 f_1,$
- (A8) $[x_2, x_1^-] \sim 0,$
- (A9) $[x_2, x_2^-] \sim 1 - g_2 f_2,$
- (A10) $[x_2, [x_1^-, x_2^-]] \sim x_1^-,$
- (A11) $[x_2, [x_2^-, x_1^-]] \sim x_1^- g_2 f_2,$
- (A12) $[x_2, [[x_1^-, x_2^-], x_2^-]] \sim [x_1^-, x_2^-],$
- (A13) $[x_2, [x_2^-, [x_2^-, x_1^-]]] \sim [x_2^-, x_1^-] g_2 f_2,$
- (A14) $[x_2, [[[x_1^-, x_2^-], x_2^-], x_2^-]] \sim [[x_1^-, x_2^-], x_2^-],$
- (A15) $[x_2, [x_2^-, [x_2^-, [x_2^-, x_1^-]]]] \sim [x_2^-, [x_2^-, x_1^-]] g_2 f_2,$
- (A16) $[x_2, [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]]] \sim [x_2^-, x_1^-]^2 + \alpha_2 [x_2^-, x_1^-]^2 g_2 f_2 + \alpha_3 [x_2^-, [x_2^-, x_1^-]] x_1^- g_2 f_2 + \alpha_4 x_2^- [x_2^-, x_1^-] x_1^- g_2 f_2,$
- (A17) $[[x_1, x_2], x_1^-] \sim g_1 f_1 x_2,$
- (A18) $[[x_1, x_2], x_2^-] \sim x_1,$
- (A19) $[[x_1, x_2], [x_2^-, x_1^-]] \sim 1 - g_1 f_1 g_2 f_2,$
- (A20) $[[x_1, x_2], [x_2^-, [x_2^-, x_1^-]]] \sim x_2^-,$
- (A21) $[[x_2, x_1], x_1^-] \sim x_2,$
- (A22) $[[x_2, x_1], x_2^-] \sim g_2 f_2 x_1,$
- (A23) $[[x_2, x_1], [x_1^-, x_2^-]] \sim 1 - g_2 f_2 g_1 f_1,$
- (A24) $[[x_2, x_1], [[x_1^-, x_2^-], x_2^-]] \sim x_2^- g_2 f_2 g_1 f_1,$
- (A25) $[[x_2, x_1], [[[x_1^-, x_2^-], x_2^-], x_2^-]] \sim (x_2^-)^2 g_2 f_2 g_1 f_1,$
- (A26) $[[x_2, x_1], [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]]] \sim [[x_1^-, x_2^-], x_2^-] g_2 f_2 g_1 f_1,$
- (A27) $[[[x_1, x_2], x_2], x_1^-] \sim g_1 f_1 x_2^2,$
- (A28) $[[[x_1, x_2], x_2], x_2^-] \sim [x_1, x_2],$
- (A29) $[[[x_1, x_2], x_2], [x_2^-, x_1^-]] \sim g_1 f_1 g_2 f_2 x_2,$
- (A30) $[[[x_1, x_2], x_2], [x_2^-, [x_2^-, x_1^-]]] \sim 1 - g_1 f_1 g_2^2 f_2^2,$
- (A31) $[[[x_1, x_2], x_2], [x_2^-, [x_2^-, [x_2^-, x_1^-]]]] \sim x_2^-,$
- (A32) $[[x_2, [x_2, x_1]], x_1^-] \sim x_2^2,$
- (A33) $[[x_2, [x_2, x_1]], x_2^-] \sim g_2 f_2 [x_2, x_1],$

- (A34) $[[x_2, [x_2, x_1]], [x_1^-, x_2^-]] \sim x_2,$
- (A35) $[[x_2, [x_2, x_1]], [[x_1^-, x_2^-], x_2^-]] \sim 1 - g_1 f_1 g_2^2 f_2^2,$
- (A36) $[[x_2, [x_2, x_1]], [[[x_1^-, x_2^-], x_2^-], x_2^-]] \sim x_2^- g_1 f_1 g_2^2 f_2^2,$
- (A37) $[[x_2, [x_2, x_1]], [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]]] \sim [x_1^-, x_2^-],$
- (A38) $[[[[x_1, x_2], x_2], x_2], x_1^-] \sim g_1 f_1 x_2^3,$
- (A39) $[[[[x_1, x_2], x_2], x_2], x_2^-] \sim [[x_1, x_2], x_2],$
- (A40) $[[[[x_1, x_2], x_2], x_2], [x_2^-, x_1^-]] \sim g_1 f_1 g_2 f_2 x_2^2,$
- (A41) $[[[[x_1, x_2], x_2], x_2], [x_2^-, [x_2^-, x_1^-]]] \sim g_1 f_1 g_2^2 f_2^2 x_2,$
- (A42) $[[[[x_1, x_2], x_2], x_2], [x_2^-, [x_2^-, [x_2^-, x_1^-]]]] \sim (1 - g_1 f_1 g_2 f_2) g_2 f_2,$
- (A43) $[[x_2, [x_2, [x_2, x_1]]], x_2^-] \sim g_2 f_2 [x_2, [x_2, x_1]],$
- (A44) $[[x_2, [x_2, [x_2, x_1]]], [[x_1^-, x_2^-], x_2^-]] \sim x_2,$
- (A45) $[[x_2, [x_2, [x_2, x_1]]], [[[x_1^-, x_2^-], x_2^-], x_2^-]] \sim 1 - g_1 f_1 g_2^3 f_2^3,$
- (A46) $[[[[x_1, x_2], [x_2, [x_2, x_1]]], x_2^-] \sim [x_2, x_1]^2 + \alpha_5 g_2 f_2 [x_2, x_1]^2 + \alpha_6 g_2 f_2 x_2 [x_2, x_1] x_1 + \alpha_7 g_2 f_2 [x_2, [x_2, x_1]] x_1,$
- (A47) $[[[x_1, x_2], [[x_1, x_2], x_2]], x_2^-] \sim [x_1, x_2]^2,$
- (A48) $[[[x_1, x_2], [[x_1, x_2], x_2]], x_1^-] \sim g_1 f_1 [[[x_1, x_2], x_2], x_2] + \alpha_8 g_1 f_1 x_2 [[x_1, x_2], x_2],$
- (A49) $[[[x_1, x_2], [[x_1, x_2], x_2]], [x_2^-, x_1^-]] \sim g_1 f_1 g_2 f_2 [[x_1, x_2], x_2],$
- (A50) $[[[x_1, x_2], [[x_1, x_2], x_2]], [x_2^-, [x_2^-, x_1^-]]] \sim [x_1, x_2].$

Now we are ready to apply Lemma 3.6 and see which spaces $\langle u^-; v \rangle$ are right coideal subalgebras and which ones are not.

Since $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ (respectively, $u_q^+(\mathfrak{g})$ and $u_q^-(\mathfrak{g})$) have 12 (homogeneous) right coideal subalgebras containing $\mathbf{k}[G]$ and $\mathbf{k}[F]$, from Lemma 3.5 we have 144 possible (homogeneous) right coideal subalgebras to study.

From the formula $ug = \chi^u(g)gu$, $g \in G$, we obtain that

$$[\mathbf{k}[G], \mathbf{U}^-] = \mathbf{k}[G]\mathbf{U}^- - \mathbf{U}^-\mathbf{k}[G] = \mathbf{U}^-\mathbf{k}[G].$$

Analogously, $[\mathbf{U}^+, \mathbf{k}[F]] = \mathbf{k}[F]\mathbf{U}^+$, and we have 23 right coideal subalgebras of the form $\mathbf{k}[F] \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \mathbf{U}^+$ or $\mathbf{U}^- \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \mathbf{k}[G]$, where \mathbf{U}^\pm runs through the set of all right coideal subalgebras of quantum Borel subalgebras.

From (A1), page 14, we get

$$[x_1, x_1^-] = 1 - g_1 f_1 \in \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle.$$

Using **(A8)** we have

$$\begin{aligned}
[x_2, x_1^-] &= 0 \in \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2], x_2 \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).
\end{aligned}$$

From **(A38)**,

$$\begin{aligned}
[[[x_1, x_2], x_2], x_2], x_1^- &\sim g_1 f_1 x_2^3 \\
&\in \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2], x_2 \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).
\end{aligned}$$

With **(A27)**,

$$\begin{aligned}
[[[x_1, x_2], x_2], x_1^-] &\sim g_1 f_1 x_2^2 \\
&\in \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).
\end{aligned}$$

Using **(A48)**,

$$\begin{aligned}
[[[x_1, x_2], [[x_1, x_2], x_2]], x_1^-] &\sim g_1 f_1 [[[x_1, x_2], x_2], x_2] + \alpha g_1 f_1 x_2 [[x_1, x_2], x_2] \\
&\in \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).
\end{aligned}$$

The equation **(A17)** provides

$$\begin{aligned}
[[x_1, x_2], x_1^-] &\sim g_1 f_1 x_2 \in \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle \\
&\subseteq \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).
\end{aligned}$$

Then, we have 7 right coideal subalgebras:

Y24. $\langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$

Y25. $\langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$

$$\mathbf{Y26.} \quad \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[x_1, x_2], x_2], x_2] \rangle,$$

$$\mathbf{Y27.} \quad \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle,$$

$$\mathbf{Y28.} \quad \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle,$$

$$\mathbf{Y29.} \quad \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle,$$

$$\mathbf{Y30.} \quad \langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).$$

Analogously, from **(A2)**, **(A9)**, **(A18)**, **(A22)**, **(A28)**, **(A33)**, **(A39)**, **(A43)** and **(A47)** we have another 6 right coideal subalgebras:

$$\mathbf{Y31.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$$

$$\mathbf{Y32.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$$

$$\mathbf{Y33.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$$

$$\mathbf{Y34.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$$

$$\mathbf{Y35.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$$

$$\mathbf{Y36.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).$$

Using **(A11)**, **(A19)**, **(A29)**, **(A40)** and **(A49)**, and the already analyzed cases, we obtain 5 more right coideal subalgebras:

$$\mathbf{Y37.} \quad \langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$$

$$\mathbf{Y38.} \quad \langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[x_1, x_2], x_2], x_2] \rangle,$$

$$\mathbf{Y39.} \quad \langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle,$$

$$\mathbf{Y40.} \quad \langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle,$$

$$\mathbf{Y41.} \quad \langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle.$$

With **(A13)**, **(A30)** and **(A41)** we also have:

$$\mathbf{Y42.} \quad \langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$$

$$\mathbf{Y43.} \quad \langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[x_1, x_2], x_2], x_2] \rangle,$$

$$\mathbf{Y44.} \quad \langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle.$$

From **(A15)** and **(A42)** we get 2 new right coideal subalgebras:

$$\mathbf{Y45.} \quad \langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$$

$$\mathbf{Y46.} \quad \langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[x_1, x_2], x_2], x_2] \rangle.$$

Using **(A6)**, **(A25)**, **(A36)** and **(A45)** we obtain the following right coideal subalgebras:

- Y47.** $\langle [[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
Y48. $\langle [[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$
Y49. $\langle [[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$
Y50. $\langle [[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle.$

With **(A5)**, **(A24)** and **(A35)** we have also:

- Y51.** $\langle [[x_1^-, x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
Y52. $\langle [[x_1^-, x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$
Y53. $\langle [[x_1^-, x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle.$

From **(A7)** and **(A26)** we get that the following possibilities are actually right coideal subalgebras:

- Y54.** $\langle [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
Y55. $\langle [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle.$

Using **(A3)** and **(A23)** we obtain:

- Y56.** $\langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
Y57. $\langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle.$

Finally, from **(A1)** to **(A3)**, **(A5)** to **(A10)**, **(A12)**, **(A14)** and **(A16)**, we have:

- Y58.** $U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
Y59. $U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle.$

We have 59 right coideal subalgebras so far. Adding the trivial case $U_q(\mathfrak{g}) = U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g})$ we complete 60. With these right coideal subalgebras we obtain the 4 suggested lattices. Notice that some right coideal subalgebras may appear on two or more of those four figures. For example, $U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g})$ is in all four figures, and $U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle$ is in two.

To complete the proof of this theorem, we still have 84 cases to consider. Now we will see that these cases are not right coideal subalgebras.

From **(A21)**, $[[x_2, x_1], x_1^-] \sim x_2$ and

$$\begin{aligned} x_2 &\notin \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle \supseteq \\ &\langle [x_2, [x_2, [x_2, x_1]]] \rangle \supseteq \\ &\langle [x_2, [x_2, x_1]] \rangle \supseteq \langle [x_2, x_1] \rangle, \end{aligned}$$

so we exclude the following 24 cases:

- N1.** $\langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$
- N2.** $\langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$
- N3.** $\langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$
- N4.** $\langle x_1^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle,$
- N5.** $\langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$
- N6.** $\langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$
- N7.** $\langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$
- N8.** $\langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle,$
- N9.** $\langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$
- N10.** $\langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$
- N11.** $\langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$
- N12.** $\langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle,$
- N13.** $\langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$
- N14.** $\langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$
- N15.** $\langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$
- N16.** $\langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle,$
- N17.** $\langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$
- N18.** $\langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$
- N19.** $\langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$
- N20.** $\langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle,$
- N21.** $U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, x_1] \rangle,$
- N22.** $U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$
- N23.** $U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$
- N24.** $U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle.$

From **(A39)**, $[[[x_1, x_2], x_2], x_2] \sim [[x_1, x_2], x_2]$. So, since we have

$$[[x_1, x_2], x_2] \notin \langle [[[[x_1, x_2], x_2], x_2] \rangle,$$

we exclude two more cases:

$$\mathbf{N25.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[[x_1, x_2], x_2], x_2]] \rangle,$$

$$\mathbf{N26.} \quad U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[[x_1, x_2], x_2], x_2]] \rangle.$$

Using **(A28)**, we obtain $[[[x_1, x_2], x_2], x_2^-] \sim [x_1, x_2]$. From the fact that

$$[x_1, x_2] \notin \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle \supseteq \langle [[x_1, x_2], x_2] \rangle,$$

we eliminate the following possibilities:

$$\mathbf{N27.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[[x_1, x_2], x_2]] \rangle,$$

$$\mathbf{N28.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[[x_1, x_2], x_2]] \rangle,$$

$$\mathbf{N29.} \quad U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle,$$

$$\mathbf{N30.} \quad U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle.$$

With **(A18)** we have $[[x_1, x_2], x_2^-] \sim x_1$, excluding:

$$\mathbf{N31.} \quad \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle,$$

$$\mathbf{N32.} \quad U_q^-(\mathfrak{g}) \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle.$$

From **(A14)**, $[x_2, [[x_1^-, x_2^-], x_2^-], x_2^-] \sim [[x_1^-, x_2^-], x_2^-]$, what eliminates:

$$\mathbf{N33.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$$

$$\mathbf{N34.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[[x_1, x_2], x_2], x_2]] \rangle,$$

$$\mathbf{N35.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle,$$

$$\mathbf{N36.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle,$$

$$\mathbf{N37.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle,$$

$$\mathbf{N38.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).$$

In **(A12)** we have $[x_2, [[x_1^-, x_2^-], x_2^-]] \sim [x_1^-, x_2^-]$, and we exclude:

$$\mathbf{N39.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$$

$$\mathbf{N40.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[[x_1, x_2], x_2], x_2]] \rangle,$$

$$\mathbf{N41.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle,$$

$$\mathbf{N42.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle,$$

$$\mathbf{N43.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle,$$

$$\mathbf{N44.} \quad \langle [[[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}),$$

$$\mathbf{N45.} \quad \langle [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$$

- N46.** $\langle [[x_1^-, x_2^-], [[[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[x_1, x_2], x_2], x_2] \rangle,$
- N47.** $\langle [[x_1^-, x_2^-], [[[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle,$
- N48.** $\langle [[x_1^-, x_2^-], [[[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle,$
- N49.** $\langle [[x_1^-, x_2^-], [[[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle,$
- N50.** $\langle [[x_1^-, x_2^-], [[[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).$

Using **(A10)**, we know that $[x_2, [x_1^-, x_2^-]] \sim x_1^-$ and eliminate 6 more possibilities:

- N51.** $\langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$
- N52.** $\langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[x_1, x_2], x_2], x_2] \rangle,$
- N53.** $\langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle,$
- N54.** $\langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [[x_1, x_2], x_2]] \rangle,$
- N55.** $\langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle,$
- N56.** $\langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).$

The equality given by **(A4)** implies $[x_1, [x_2^-, x_1^-]] \sim x_2^-$, excluding the cases:

- N57.** $\langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
- N58.** $\langle [x_2^-, x_1^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}),$
- N59.** $\langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
- N60.** $\langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}),$
- N61.** $\langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
- N62.** $\langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}),$
- N63.** $\langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_1 \rangle,$
- N64.** $\langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_q^+(\mathfrak{g}).$

From **(A44)** we know that $[[x_2, [x_2, [x_2, x_1]]], [[x_1^-, x_2^-], x_2^-]] \sim x_2$, what eliminates:

- N65.** $\langle [[x_1^-, x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$
- N66.** $\langle [[x_1^-, x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle.$

With **(A37)** we have $[[x_2, [x_2, x_1]], [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]]] \sim [x_1^-, x_2^-]$, and we obtain that

$$\mathbf{N67.} \langle [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$$

$$\mathbf{N68.} \langle [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$$

$$\mathbf{N69.} \langle [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle,$$

are not right coideal subalgebras.

From the equality in **(A34)**, $[[x_2, [x_2, x_1]], [x_1^-, x_2^-]] \sim x_2$, and the following cases are excluded:

$$\mathbf{N70.} \langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, x_1]] \rangle,$$

$$\mathbf{N71.} \langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_2, [x_2, [x_2, x_1]]] \rangle,$$

$$\mathbf{N72.} \langle [x_1^-, x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle.$$

In **(A31)** we have $[[[x_1, x_2], x_2], [x_2^-, [x_2^-, [x_2^-, x_1^-]]]] \sim x_2^-$, eliminating the cases:

$$\mathbf{N73.} \langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle,$$

$$\mathbf{N74.} \langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], x_2] \rangle.$$

Using **(A50)** we obtain $[[[x_1, x_2], [x_1, x_2], x_2], [x_2^-, [x_2^-, x_1^-]]] \sim [x_1, x_2]$, what excludes:

$$\mathbf{N75.} \langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_1, x_2], x_2] \rangle,$$

$$\mathbf{N76.} \langle [[x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_1, x_2], x_2] \rangle,$$

$$\mathbf{N77.} \langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_1, x_2], x_2] \rangle.$$

With **(A20)**, we have $[[x_1, x_2], [x_2^-, [x_2^-, x_1^-]]] \sim x_2^-$, and we eliminate:

$$\mathbf{N78.} \langle [x_2^-, [x_2^-, x_1^-]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle,$$

$$\mathbf{N79.} \langle [x_2^-, [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle,$$

$$\mathbf{N80.} \langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [x_1, x_2] \rangle.$$

For the last 4 cases we are going to use Proposition 2.6. From **(A46)**, $[[[x_1, x_2], [x_2, [x_2, x_1]]], x_2^-] \sim [x_2, x_1]^2 + \alpha_1 g_2 f_2 x_2 [x_2, x_1] x_1 + \alpha_2 g_2 f_2 [x_2, [x_2, x_1]] x_1 + \alpha_3 g_2 f_2 [x_2, x_1]^2$. Since $x_2 \notin \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle$, using the proposition, we exclude:

$$\mathbf{N81.} \langle x_2^- \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle,$$

$$\mathbf{N82.} \langle [[[x_1^-, x_2^-], x_2^-], x_2^-] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[x_1, x_2], [x_2, [x_2, x_1]]] \rangle.$$

Analogously, from **(A16)** we have $[x_2, [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]]] \sim [x_2^-, x_1^-]^2 + \alpha_1 [x_2^-, x_1^-]^2 g_2 f_2 + \alpha_2 [x_2^-, [x_2^-, x_1^-]] x_1^- g_2 f_2 + \alpha_3 x_2^- [x_2^-, x_1^-] x_1^- g_2 f_2$. As we know that $x_2^- \notin \langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle$, again from Proposition 2.6 we eliminate the final possibilities:

$$\mathbf{N83.} \quad \langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle x_2 \rangle,$$

$$\mathbf{N84.} \quad \langle [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]] \rangle \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} \langle [[[x_1, x_2], x_2], x_2] \rangle.$$

Now we have showed that the first 60 cases are the only (homogeneous) right coideal subalgebras of $U_q(\mathfrak{g})$ (respectively, $u_q(\mathfrak{g})$) that contain $\mathbf{k}[H]$. The theorem is proved. \square

Appendix

In this appendix we are going to calculate the skew commutators that are listed in the previous section. Below by u, v, w we denote homogeneous linear combinations of words in x_1, x_2, \dots, x_n , while u^-, v^-, w^- are the elements that appeared from u, v, w by replacing all x_i with x_i^- .

We notice that, from Definition 2.10, we have:

$$p(u, v) = p_{uv}, \quad p(u, v^-) = p_{vu}, \quad p(u^-, v) = p_{uv}^{-1}, \quad p(u^-, v^-) = p_{vu}^{-1}.$$

We also have the following relations, that are a consequence of (8) and (9):

$$\begin{aligned} x_1 g_1 &= p_{11} g_1 x_1, & x_1 g_2 &= p_{12} g_2 x_1, & x_1 f_1 &= p_{11} f_1 x_1, & x_1 f_2 &= p_{21} f_2 x_1, \\ x_2 g_1 &= p_{21} g_1 x_2, & x_2 g_2 &= p_{22} g_2 x_2, & x_2 f_1 &= p_{12} f_1 x_2, & x_2 f_2 &= p_{22} f_2 x_2, \\ g_1 x_1^- &= p_{11} x_1^- g_1, & g_2 x_1^- &= p_{12} x_1^- g_2, & f_1 x_1^- &= p_{11} x_1^- f_1, & f_2 x_1^- &= p_{21} x_1^- f_2, \\ g_1 x_2^- &= p_{21} x_2^- g_1, & g_2 x_2^- &= p_{22} x_2^- g_2, & f_1 x_2^- &= p_{12} x_2^- f_1, & f_2 x_2^- &= p_{22} x_2^- f_2. \end{aligned}$$

where $p_{22} = q$, $p_{11} = q^3$, $p_{12} p_{21} = q^{-3}$.

For simplification, we are also going to use the formulas, see [7] page 2612:

$$[[u, v], w^-] = [u, [v, w^-]] + p_{wv} [[u, w^-], v],$$

$$[u, [v^-, w^-]] = [[u, v^-], w^-] + p_{vu} [v^-, [u, w^-]],$$

$$[u \cdot v, w] = p_{vw} [u, w] \cdot v + u \cdot [v, w],$$

$$[u, v \cdot w] = [u, v] \cdot w + p_{uv} v \cdot [u, w].$$

Now we are ready to calculate the skew-commutators.

$$(\mathbf{A1}) \quad [x_1, x_1^-] = 1 - g_1 f_1,$$

$$(\mathbf{A2}) \quad [x_1, x_2^-] = 0,$$

$$\begin{aligned}
(\mathbf{A3}) \quad & [x_1, [x_1^-, x_2^-]] = [[x_1, x_1^-], x_2^-] + p_{11}[x_1^-, [x_1, x_2^-]] = (1 - g_1 f_1)x_2^- - x_2^-(1 - g_1 f_1) = -g_1 f_1 x_2^- + x_2^- g_1 f_1 = -p_{12}p_{21}x_2^- g_1 f_1 + x_2^- g_1 f_1 = (1 - q^{-3})x_2^- g_1 f_1, \\
(\mathbf{A4}) \quad & [x_1, [x_2^-, x_1^-]] = [x_1, x_2^- x_1^-] - p_{12}^{-1}[x_1, x_1^- x_2^-] = [x_1, x_2^-]x_1^- + p_{21}x_2^- [x_1, x_1^-] - p_{12}^{-1}[x_1, x_1^-]x_2^- - p_{12}^{-1}p_{11}x_1^- [x_1, x_2^-] = p_{21}x_2^- (1 - g_1 f_1) - p_{12}^{-1}(1 - g_1 f_1)x_2^- = p_{21}x_2^- - p_{21}x_2^- g_1 f_1 - p_{12}^{-1}x_2^- + p_{12}^{-1}p_{12}p_{21}x_2^- g_1 f_1 = (p_{21} - p_{12}^{-1})x_2^- = p_{21}(1 - q^3)x_2^-, \\
(\mathbf{A5}) \quad & [x_1, [[x_1^-, x_2^-], x_2^-]] = [[x_1, [x_1^-, x_2^-]], x_2^-] + p_{11}p_{21}[[x_1^-, x_2^-], [x_1, x_2^-]] = (1 - q^{-3})x_2^- g_1 f_1 x_2^- - p_{22}^{-1}x_2^- (1 - q^{-3})x_2^- g_1 f_1 = (1 - q^{-3})(p_{12}p_{21} - p_{22}^{-1})(x_2^-)^2 g_1 f_1 = -q^{-1}(1 - q^{-2})(1 - q^{-3})(x_2^-)^2 g_1 f_1, \\
(\mathbf{A6}) \quad & [x_1, [[[x_1^-, x_2^-], x_2^-], x_2^-]] = [[x_1, [[x_1^-, x_2^-], x_2^-]], x_2^-] + p_{11}p_{21}^2[[[x_1^-, x_2^-], x_2^-], [x_1, x_2^-]] = -q^{-1}(1 - q^{-3})(1 - q^{-2})(x_2^-)^2 g_1 f_1 x_2^- + q^{-2}q^{-1}(1 - q^{-3})(1 - q^{-2})(x_2^-)^3 g_1 f_1 = q^{-3}(1 - q^{-1})(1 - q^{-2})(1 - q^{-3})(x_2^-)^3 g_1 f_1, \\
(\mathbf{A7}) \quad & [x_1, [[x_1^-, x_2^-], [[x_1^-, x_2^-], x_2^-]]] = [[x_1, [x_1^-, x_2^-]], [[x_1^-, x_2^-], x_2^-]] + p_{11}p_{21}[[x_1^-, x_2^-], [x_1, [[x_1^-, x_2^-], x_2^-]]] = (1 - q^{-3})x_2^- g_1 f_1 [[x_1^-, x_2^-], x_2^-] - (1 - q^{-3})p_{11}p_{21}^2 p_{11}^{-1} p_{12}^{-2} p_{12}^{-1} p_{22}^{-2} [[x_1^-, x_2^-], x_2^-] x_2^- g_1 f_1 - p_{11}p_{21}q^{-1}(1 - q^{-3})(1 - q^{-2})[x_1^-, x_2^-](x_2^-)^2 g_1 f_1 + p_{11}p_{21}p_{11}^{-1} p_{21}^{-1} p_{21}^{-1} p_{22}^{-2} q^{-1}(1 - q^{-3})(1 - q^{-2})(x_2^-)^2 g_1 f_1 [x_1^-, x_2^-] = (1 - q^{-3})q^6 q^{-6} x_2^- [[x_1^-, x_2^-], x_2^-] g_1 f_1 - p_{21}q(1 - q^{-3})([[x_1^-, x_2^-], x_2^-], x_2^-] + p_{21}^{-1}q^{-2}x_2^- [[x_1^-, x_2^-], x_2^-] g_1 f_1 - p_{21}q^2(1 - q^{-3})(1 - q^{-2})([[x_1^-, x_2^-], x_2^-], x_2^-] + p_{12}q(1 + q)x_2^- [[x_1^-, x_2^-], x_2^-] + p_{12}^2 q^4 (x_2^-)^2 [x_1^-, x_2^-] g_1 f_1 + p_{12}(1 - q^{-3})(1 - q^{-2})q^6 q^{-3}(x_2^-)^2 [x_1^-, x_2^-] g_1 f_1 = (1 - q^{-3})(1 - q^{-1} - (1 - q^{-2})(1 + q))x_2^- [[x_1^-, x_2^-], x_2^-] g_1 f_1 - p_{21}q(1 - q^{-3})(1 + q(1 - q^{-2}))[[[x_1^-, x_2^-], x_2^-], x_2^-] g_1 f_1 + p_{12}(1 - q^{-3})(1 - q^{-2})(q^3 - q^3)(x_2^-)^2 [x_1^-, x_2^-] g_1 f_1 = -q(1 - q^{-3})^2 x_2^- [[x_1^-, x_2^-], x_2^-] g_1 f_1 - p_{21}q(1 - q^{-3})(1 + q - q^{-1})[[[x_1^-, x_2^-], x_2^-], x_2^-] g_1 f_1, \\
(\mathbf{A8}) \quad & [x_2, x_1^-] = 0, \\
(\mathbf{A9}) \quad & [x_2, x_2^-] = 1 - g_2 f_2, \\
(\mathbf{A10}) \quad & [x_2, [x_1^-, x_2^-]] = p_{12}(1 - q^3)x_1^-, \\
(\mathbf{A11}) \quad & [x_2, [x_2^-, x_1^-]] = (1 - q^{-3})x_1^- g_2 f_2, \\
(\mathbf{A12}) \quad & [x_2, [[x_1^-, x_2^-], x_2^-]] = [x_2, [x_1^-, x_2^-]x_2^-] - p_{21}^{-1}p_{22}^{-1}[x_2, x_2^- [x_1^-, x_2^-]] = [x_2, [x_1^-, x_2^-]]x_2^- + p_{12}p_{22}[x_1^-, x_2^-][x_2, x_2^-] - p_{21}^{-1}p_{22}^{-1}[x_2, x_2^-][x_1^-, x_2^-] - p_{21}^{-1}p_{22}^{-1}p_{22}x_2^- [x_2, [x_1^-, x_2^-]] = p_{12}(1 - q^3)x_1^- x_2^- + p_{12}p_{22}[x_1^-, x_2^-](1 - g_2 f_2) - p_{21}^{-1}p_{22}^{-1}(1 - g_2 f_2)[x_1^-, x_2^-] - p_{22}^{-1}p_{21}^{-1}p_{22}x_2^- p_{12}(1 - q^3)x_1^- = p_{12}(1 - q^3)[x_1^-, x_2^-] + p_{12}q[x_1^-, x_2^-] - p_{12}q^2[x_1^-, x_2^-] - p_{12}q[x_1^-, x_2^-]g_2 f_2 + p_{12}q^2 g_2 f_2 [x_1^-, x_2^-] = (p_{12}q^4 q^{-3} - p_{12}q)g_2 f_2 [x_1^-, x_2^-] + p_{12}(1 + q - q^2 - q^3)[x_1^-, x_2^-] = p_{12}(1 + q)(1 - q^2)[x_1^-, x_2^-], \\
(\mathbf{A13}) \quad & [x_2, [x_2^-, [x_2^-, x_1^-]]] = [x_2, x_2^- [x_2^-, x_1^-]] - p_{22}^{-1}p_{12}^{-1}[x_2, [x_2^-, x_1^-]x_2^-] = [x_2, x_2^-][x_2^-, x_1^-] + p_{22}x_2^- [x_2, [x_2^-, x_1^-]] - p_{22}^{-1}p_{12}^{-1}[x_2, [x_2^-, x_1^-]]x_2^- - p_{22}^{-1}p_{12}^{-1}p_{22}p_{12}[x_2^-, x_1^-][x_2, x_2^-] = (1 - g_2 f_2)[x_2^-, x_1^-] + p_{22}x_2^- (1 - q^{-3})x_1^- g_2 f_2 - p_{22}^{-1}p_{12}^{-1}(1 - q^{-3})x_1^- g_2 f_2 x_2^- - [x_2^-, x_1^-](1 - g_2 f_2) = [x_2^-, x_1^-] - g_2 f_2 [x_2^-, x_1^-] + p_{22}(1 - q^{-3})x_2^- x_1^- g_2 f_2 - p_{22}p_{12}^{-1}(1 - q^{-3})x_1^- x_2^- g_2 f_2 - [x_2^-, x_1^-] + [x_2^-, x_1^-]g_2 f_2 = (1 - q^{-1})[x_2^-, x_1^-]g_2 f_2 + q(1 - q^{-3})[x_2^-, x_1^-]g_2 f_2 = (1 + q)(1 - q^{-2})[x_2^-, x_1^-]g_2 f_2,
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A14}) \quad & [x_2, [[[x_1^-, x_2^-], x_2^-], x_2^-]] = [x_2, [[x_1^-, x_2^-], x_2^-]x_2^-] - p_{22}^{-2}p_{21}^{-1}[x_2, x_2^-][[x_1^-, x_2^-], x_2^-] = \\
& [x_2, [[x_1^-, x_2^-], x_2^-]x_2^- + p_{12}p_{22}^2[[x_1^-, x_2^-], x_2^-][x_2, x_2^-] - p_{22}^{-2}p_{21}^{-1}[x_2, x_2^-][[x_1^-, x_2^-], x_2^-] - \\
& p_{22}^{-2}p_{21}^{-1}p_{22}x_2^-][x_2, [[x_1^-, x_2^-], x_2^-]] = p_{12}(1+q-q^2-q^3)[x_1^-, x_2^-]x_2^- + p_{12}p_{22}^2[[x_1^-, x_2^-], x_2^-] \\
& (1 - g_2f_2) - p_{22}^{-2}p_{21}^{-1}(1 - g_2f_2)[[x_1^-, x_2^-], x_2^-] - p_{21}^{-1}p_{22}^{-1}p_{12}(1 + q - q^2 - \\
& q^3)x_2^-[x_1^-, x_2^-] = p_{12}(1+q-q^2-q^3)[[x_1^-, x_2^-], x_2^-] + (p_{12}q^2 - p_{12}q)[[x_1^-, x_2^-], x_2^-] - \\
& (p_{12}q^2 - p_{12}q^2)[[x_1^-, x_2^-], x_2^-]g_2f_2 = p_{12}(1 - q^3)[[x_1^-, x_2^-], x_2^-],
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A15}) \quad & [x_2, [x_2^-, [x_2^-, [x_2^-, x_1^-]]]] = [x_2, x_2^-[x_2^-, [x_2^-, x_1^-]]] - p_{22}^{-2}p_{12}^{-1}[x_2, [x_2^-, [x_2^-, x_1^-]]x_2^-] = \\
& [x_2, x_2^-][x_2^-, [x_2^-, x_1^-]] + p_{22}x_2^-[x_2, [x_2^-, [x_2^-, x_1^-]]] - p_{22}^{-2}p_{12}^{-1}[x_2, [x_2^-, [x_2^-, x_1^-]]]x_2^- - \\
& p_{22}^{-2}p_{12}^{-1}p_{22}^2p_{12}[x_2^-, [x_2^-, x_1^-]][x_2, x_2^-] = (1 - g_2f_2)[x_2^-, [x_2^-, x_1^-]] + p_{22}(1 + q - \\
& q^{-1} - q^{-2})x_2^-[x_2^-, x_1^-]g_2f_2 - p_{22}^{-2}p_{12}^{-1}(1 + q - q^{-1} - q^{-2})[x_2^-, x_1^-]g_2f_2x_2^- - \\
& [x_2^-, [x_2^-, x_1^-]](1 - g_2f_2) = p_{22}(1 + q - q^{-1} - q^{-2})[x_2^-, [x_2^-, x_1^-]]g_2f_2 = \\
& q^2(1 - q^{-3})[x_2^-, [x_2^-, x_1^-]]g_2f_2,
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A16}) \quad & [x_2, [[x_1^-, x_2^-], [x_2^-, [x_2^-, x_1^-]]]] = [[x_2, [x_1^-, x_2^-]], [x_2^-, [x_2^-, x_1^-]]] + p_{12}p_{22}[[x_1^-, x_2^-], \\
& [x_2, [x_2^-, [x_2^-, x_1^-]]]] = p_{12}(1 - q^3)x_1^-[x_2^-, [x_2^-, x_1^-]] - p_{22}^2p_{12}p_{21}^{-1}p_{22}^{-2}p_{11}^{-1}p_{12}^{-1}p_{12}(1 - \\
& q^3)[x_2^-, [x_2^-, x_1^-]]x_1^- + p_{12}p_{22}(1 + q - q^{-1} - q^{-2})[x_1^-, x_2^-][x_2^-, x_1^-]g_2f_2 - p_{12}p_{22}^{-1}p_{21}^{-1}p_{22}^{-2}p_{11}^{-1}p_{12}^{-1}p_{12}p_{22}(1 + \\
& q - q^{-1} - q^{-2})[x_2^-, x_1^-]g_2f_2[x_1^-, x_2^-] = p_{12}(1 - q^3)(p_{12}q^2(1 - q^{-2})[x_2^-, x_1^-]^2 + \\
& p_{12}^2q^3[x_2^-, [x_2^-, x_1^-]]x_1^-) - p_{12}^3q^3(1 - q^3)[x_2^-, [x_2^-, x_1^-]]x_1^- + p_{12}q(1 + q - q^{-1} - \\
& q^{-2})(p_{12}(1 - q^3)x_2^-x_1^- - p_{12}[x_2^-, x_1^-])[x_2^-, x_1^-]g_2f_2 - p_{12}(1 + q - q^{-1} - q^{-2})[x_2^-, x_1^-]q^{-1}(p_{12}(1 - \\
& q^3)x_2^-x_1^- - p_{12}[x_2^-, x_1^-])g_2f_2 = p_{12}^2q^2(1 - q^3)(1 - q^{-2})[x_2^-, x_1^-]^2 - p_{12}^2q(1 + \\
& q)(1 - q^{-2})^2[x_2^-, x_1^-]^2g_2f_2 - p_{12}^3(1 + q)(1 - q^3)(1 - q^{-2})[x_2^-, [x_2^-, x_1^-]]x_1^-g_2f_2 - \\
& p_{12}^3(1 + q)(1 - q^3)(1 - q^4)(1 - q^{-2})x_2^-[x_2^-, x_1^-]x_1^-g_2f_2,
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A17}) \quad & [[x_1, x_2], x_1^-] = [x_1x_2, x_1^-] - p_{12}[x_2x_1, x_1^-] = p_{12}[x_1, x_1^-]x_2 + x_1[x_2, x_1^-] - \\
& p_{12}p_{11}[x_2, x_1^-]x_1 - p_{12}x_2[x_1, x_1^-] = p_{12}(1 - g_1f_1)x_2 - p_{12}x_2(1 - g_1f_1) = \\
& p_{12}^2p_{21}g_1f_1x_2 - p_{12}g_1f_1x_2 = -p_{12}(1 - q^{-3})g_1f_1x_2,
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A18}) \quad & [[x_1, x_2], x_2^-] = [x_1x_2, x_2^-] - p_{12}[x_2x_1, x_2^-] = p_{22}[x_1, x_2^-]x_2 + x_1[x_2, x_2^-] - \\
& p_{12}p_{21}[x_2, x_2^-]x_1 - p_{12}x_2[x_1, x_2^-] = x_1(1 - g_2f_2) - p_{12}p_{21}(1 - g_2f_2)x_1 = \\
& (1 - q^{-3})x_1,
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A19}) \quad & [[x_1, x_2], [x_2^-, x_1^-]] = [x_1x_2, [x_2^-, x_1^-]] - p_{12}[x_2x_1, [x_2^-, x_1^-]] = p_{12}p_{22}[x_1, [x_2^-, x_1^-]]x_2 + \\
& x_1[x_2, [x_2^-, x_1^-]] - p_{12}p_{11}p_{21}[x_2, [x_2^-, x_1^-]]x_1 - p_{12}x_2[x_1, [x_2^-, x_1^-]] = p_{12}p_{21}q(1 - \\
& q^3)x_2^-x_2 + (1 - q^{-3})x_1x_1^-g_2f_2 - (1 - q^{-3})x_1^-g_2f_2x_1 - p_{12}p_{21}q(1 - q^3)x_2x_2^- = \\
& (q^3 - 1)x_1^-x_1g_2f_2 + (1 - q^{-3})g_2f_2 - (1 - q^{-3})g_1f_1g_2f_2 - (1 - q^{-3})x_1^-g_2f_2x_1 - \\
& q^{-3}(1 - q^3) + q^{-3}(1 - q^3)g_2f_2 = (1 - q^{-3})(1 - g_1f_1g_2f_2),
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A20}) \quad & [[x_1, x_2], [x_2^-, [x_2^-, x_1^-]]] = [[[x_1, x_2], x_2^-], [x_2^-, x_1^-]] + p_{21}p_{22}[x_2^-, [[x_1, x_2], [x_2^-, x_1^-]]] = \\
& (1 - q^{-3})x_1[x_2^-, x_1^-] - p_{21}p_{22}p_{11}p_{12}p_{22}^{-1}p_{12}^{-1}(1 - q^{-3})[x_2^-, x_1^-]x_1 + p_{21}p_{22}(1 - \\
& q^{-3})x_2^-(1 - g_1f_1g_2f_2) - p_{21}p_{22}(1 - q^{-3})p_{21}^{-1}p_{22}^{-1}p_{22}^{-1}p_{12}^{-1}(1 - g_1f_1g_2f_2)x_2^- = \\
& (1 - q^{-3})[x_1, [x_2^-, x_1^-]] + p_{21}p_{22}(1 - q^{-3})(1 - q)x_2^- = p_{21}(1 - q^{-3})(1 - \\
& q^{-3})x_2^- + p_{21}p_{22}q(1 - q^{-3})(1 - q)x_2^- = p_{21}(1 + q)(1 - q^2)(1 - q^{-3})x_2^-,
\end{aligned}$$

$$(\mathbf{A21}) \quad [[x_2, x_1], x_1^-] = (1 - q^{-3})x_2,$$

$$(\mathbf{A22}) \quad [[x_2, x_1], x_2^-] = -p_{21}(1 - q^{-3})g_2f_2x_1,$$

$$(\mathbf{A23}) \quad [[x_2, x_1], [x_1^-, x_2^-]] = (1 - q^{-3})(1 - g_2f_2g_1f_1),$$

$$\begin{aligned}
\text{(A24)} \quad & [[x_2, x_1], [x_1^-, x_2^-], x_2^-] = [[[x_2, x_1], [x_1^-, x_2^-]], x_2^-] + p_{12}p_{11}p_{22}p_{21}[[x_1^-, x_2^-], [[x_2, x_1], x_2^-]] = \\
& (1 - q^{-3})(1 - g_2f_2g_1f_1)x_2^- - p_{22}p_{21}p_{21}^{-1}p_{22}^{-1}(1 - q^{-3})x_2^-(1 - g_2f_2g_1f_1) - \\
& qp_{21}(1 - q^{-3})[x_1^-, x_2^-]g_2f_2x_1 + qp_{21}(1 - q^{-3})p_{21}^{-1}p_{22}^{-1}p_{12}^{-1}p_{11}^{-1}p_{22}^{-1}p_{21}^{-1}g_2f_2x_1[x_1^-, x_2^-] = \\
& (1 - q^{-3})(1 - q^{-1})x_2^-g_2f_2g_1f_1 - p_{21}q(1 - q^{-3})[x_1^-, x_2^-]g_2f_2x_1 + q^{-1}(1 - q^{-3})g_2f_2((1 - q^{-3})x_2^-g_1f_1 + p_{11}p_{21}[x_1^-, x_2^-]x_1) = (1 - q^{-3})(1 - q^{-1})x_2^-g_2f_2g_1f_1 - \\
& p_{21}q(1 - q^{-3})[x_1^-, x_2^-]g_2f_2x_1 + q(1 - q^{-3})^2x_2^-g_2f_2g_1f_1 + p_{21}q^2(1 - q^{-3})p_{12}p_{21}q^2 \\
& [x_1^-, x_2^-]g_2f_2x_1 = (1 - q)(1 - q^{-2})(1 - q^{-3})x_2^-g_1f_1g_2f_2, \\
\text{(A25)} \quad & [[x_2, x_1], [[x_1^-, x_2^-], x_2^-]] = [[[x_2, x_1], [x_1^-, x_2^-], x_2^-], x_2^-] + p_{12}p_{11}p_{22}^2p_{21}^2 \\
& [[[x_1^-, x_2^-], x_2^-], [[x_2, x_1], x_2^-]] = (1 - q^{-3})(1 - q^{-2})(1 + q)x_2^-g_1f_1g_2f_2x_2^- - \\
& p_{22}p_{21}p_{21}^{-1}p_{22}^{-2}(1 - q^{-3})(1 - q^{-2})(1 + q)(x_2^-)^2g_1f_1g_2f_2 - p_{21}^2q^2[[x_1^-, x_2^-], x_2^-](1 - q^{-3}) \\
& g_2f_2x_1 + p_{21}^2q^2(1 - q^{-3})p_{12}^{-1}p_{11}^{-1}p_{22}^{-2}p_{21}^{-1}p_{22}^{-2}g_2f_2x_1[[x_1^-, x_2^-], x_2^-] = \\
& -p_{21}^2q^2(1 - q^{-3})[[x_1^-, x_2^-], x_2^-]g_2f_2x_1 + q^{-2}(1 - q^{-3})g_2f_2(p_{21}^2p_{11}[[x_1^-, x_2^-], x_2^-]x_1 - \\
& q^{-1}(1 - q^{-3})(1 - q^{-2})(x_2^-)^2g_1f_1) = -q(1 - q^{-2})(1 - q^{-3})^2(x_2^-)^2g_1f_1g_2f_2, \\
\text{(A26)} \quad & [[x_2, x_1], [[x_1^-, x_2^-], [x_1^-, x_2^-], x_2^-]] = [[[x_2, x_1], [x_1^-, x_2^-], [x_1^-, x_2^-], x_2^-]] + \\
& p_{12}p_{11} \times p_{22}p_{21}[[x_2^-, x_1^-], [[x_1^-, x_2^-], [x_1^-, x_2^-], x_2^-]] = (1 - q^{-3})(1 - g_2f_2g_1f_1)[[x_1^-, x_2^-], x_2^-] - \\
& p_{12}p_{11}p_{22}^2p_{21}^2p_{11}^{-1}p_{12}^{-2}p_{21}^{-2}p_{22}^{-2}(1 - q^{-3})[[x_1^-, x_2^-], x_2^-](1 - g_2f_2g_1f_1) + p_{22}(1 - q^{-3}) \\
& (1 - q^{-2})(1 + q)[x_1^-, x_2^-]x_2^-g_2f_2g_1f_1 - p_{22}(1 - q^{-3})(1 - q^{-2})(1 + q)p_{12}^{-2}p_{11}^{-2}p_{22}^{-2}p_{21}^{-2} \times \\
& x_2^-g_2f_2g_1f_1[x_1^-, x_2^-] = (1 - q)(1 - q^{-3})[[x_1^-, x_2^-], x_2^-]g_1f_1g_2f_2 + p_{22}(1 - q^{-3}) \\
& (1 - q^{-2})(1 + q)[x_1^-, x_2^-]x_2^-g_2f_2g_1f_1 - p_{12}p_{22}^3(1 - q^{-3})(1 - q^{-2})(1 + q)x_2^-[x_1^-, x_2^-]g_2f_2g_1f_1 = \\
& (1 - q)(1 - q^{-3})[[x_1^-, x_2^-], x_2^-]g_1f_1g_2f_2 + p_{22}(1 - q^{-3})(1 - q^{-2})(1 + q)[[x_1^-, x_2^-], x_2^-]g_2f_1g_1f_1 = (1 - q^{-3})(1 - q + q(1 - q^{-2}))(1 + q) \\
& [[x_1^-, x_2^-], x_2^-]g_2f_2g_1f_1 = q^2(1 - q^{-3})[[x_1^-, x_2^-], x_2^-]g_1f_1g_2f_2, \\
\text{(A27)} \quad & [[[x_1, x_2], x_2], x_1^-] = [[x_1, x_2]x_2, x_1^-] - p_{12}p_{22}[x_2[x_1, x_2], x_1^-] = p_{12}[[x_1, x_2], x_1^-]x_2 + \\
& [x_1, x_2][x_2, x_1^-] - p_{12}p_{22}p_{11}p_{12}[x_2, x_1^-][x_1, x_2] - p_{12}p_{22}x_2[[x_1, x_2], x_1^-] = -p_{12}^2(1 - q^{-3})g_1f_1x_2^2 + \\
& p_{12}^3p_{21}p_{22}(1 - q^{-3})g_1f_1x_2^2 = -p_{12}^2(1 - q^{-2})(1 - q^{-3})g_1f_1x_2^2, \\
\text{(A28)} \quad & [[[x_1, x_2], x_2], x_2^-] = [[x_1, x_2]x_2, x_2^-] - p_{12}p_{22}[x_2[x_1, x_2], x_2^-] = p_{22}[[x_1, x_2], x_2^-]x_2 + \\
& [x_1, x_2][x_2, x_2^-] - p_{12}p_{22}p_{21}p_{22}[x_2, x_2^-][x_1, x_2] - p_{12}p_{22}x_2[[x_1, x_2], x_2^-] = p_{22}(1 - q^{-3})x_1x_2 + \\
& [x_1, x_2](1 - g_2f_2) - q^{-1}(1 - g_2f_2) - p_{12}p_{22}(1 - q^{-3})x_2x_1 = (1 + q)(1 - q^{-2})[x_1, x_2], \\
\text{(A29)} \quad & [[[x_1, x_2], x_2], [x_2^-, x_1^-]] = [[x_1, x_2], [x_2, [x_2^-, x_1^-]]] + p_{22}p_{12}[[x_1, x_2], [x_2^-, x_1^-], x_2] = \\
& (1 - q^{-3})[x_1, x_2]x_1^-g_2f_2 - p_{12}p_{22}p_{21}p_{22}p_{11}p_{12}(1 - q^{-3})x_1^-g_2f_2[x_1, x_2] + p_{12}p_{22}(1 - q^{-3}) \\
& (1 - g_1f_1g_2f_2)x_2 - p_{12}p_{22}p_{12}p_{22}p_{22}^{-1}p_{12}^{-1}(1 - q^{-3})x_2(1 - g_1f_1g_2f_2) = (1 - q^{-3})(p_{12}p_{11}x_1^-[x_1, x_2] - p_{12}(1 - q^{-3})g_1f_1x_2)g_2f_2 - p_{12}q^2(1 - q^{-3})x_1^-g_2f_2[x_1, x_2] - \\
& (1 - q^{-1})p_{12}q(1 - q^{-3})g_1f_1g_2f_2x_2 = -p_{12}(1 - q^{-3})^2q^2g_1f_1g_2f_2x_2 - p_{12}q(1 - q^{-1})(1 - q^{-3})g_1f_1g_2f_2x_2 = -p_{12}q(1 + q)(1 - q^{-2})(1 - q^{-3})g_1f_1g_2f_2x_2, \\
\text{(A30)} \quad & [[[x_1, x_2], x_2], [x_2^-, [x_1^-, x_2^-]]] = [[[x_1, x_2], x_2], [x_2^-, x_1^-]] + p_{21}q^2[x_2^-, [[x_1, x_2], x_2]] = \\
& (1 + q - q^{-1} - q^{-2})[x_1, x_2][x_2^-, x_1^-] - p_{22}^{-1}p_{12}^{-1}p_{21}p_{22}^2p_{11}p_{12}^2(1 + q - q^{-1} - q^{-2})[x_2^-, x_1^-][x_1, x_2] - p_{21}q^2p_{12}q(1 - q^{-3})(1 - q^{-2})(1 + q)x_2^-g_1f_1g_2f_2x_2 + \\
& (1 - q^{-3})(1 - q^{-2})(1 + q)p_{22}^{-1}p_{12}^{-1}p_{21}p_{22}^2p_{11}p_{12}^2g_1f_1g_2f_2x_2x_2^- = (1 + q - q^{-1} - q^{-2})((1 - q^{-3})(1 - g_1f_1g_2f_2) + q[x_2^-, x_1^-][x_1, x_2]) - q(1 + q - q^{-1} - q^{-2})[x_2^-, x_1^-][x_1, x_2] - (1 - q^{-3})(1 - q^{-2})(1 + q)x_2^-g_1f_1g_2f_2x_2 + (1 - q^{-3})(1 - q^{-2})(1 + q)g_1f_1g_2f_2(1 - g_2f_2 + qx_2^-x_2) = (1 + q - q^{-1} - q^{-2})(1 - q^{-3})(1 - q^{-2})(1 + q)g_1f_1g_2f_2(1 - g_2f_2 + qx_2^-x_2)
\end{aligned}$$

$$g_1 f_1 g_2 f_2) - (1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- g_1 f_1 g_2 f_2 x_2 + (1 - q^{-3})(1 - q^{-2})(1 + q)g_1 f_1 g_2 f_2(1 - g_2 f_2) + (1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- g_1 f_1 g_2 f_2 x_2 = (1 + q)(1 - q^{-2})(1 - q^{-3})(1 - g_1 f_1 g_2^2 f_2^2),$$

$$\begin{aligned} \text{(A31)} \quad & [[[x_1, x_2], x_2], [x_2^-, [x_2^-, [x_2^-, x_1^-]]]] = [[[x_1, x_2], x_2], x_2^-], [x_2^-, [x_2^-, x_1^-]] + p_{21} q^2 \\ & [x_2^-, [[[x_1, x_2], x_2], [x_2^-, [x_2^-, x_1^-]]]] = (1 + q - q^{-1} - q^{-2})[x_1, x_2][x_2^-, [x_2^-, x_1^-]] - \\ & p_{21}^2 p_{22}^4 p_{11} p_{12}^2 p_{22}^2 p_{12}^{-1} (1 + q - q^{-1} - q^{-2})[x_2^-, [x_2^-, x_1^-]][x_1, x_2] + p_{21} q^2 (1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- (1 - g_1 f_1 g_2^2 f_2^2) - p_{21} q^2 p_{21}^{-1} p_{22}^2 p_{22}^2 p_{12}^{-1} (1 - q^{-3})(1 - q^{-2})(1 + q)(1 - g_1 f_1 g_2^2 f_2^2)x_2^- = (1 + q - q^{-1} - q^{-2})(p_{21}(1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- + p_{21}^2 p_{22}^2 p_{11} p_{12}[x_2^-, [x_2^-, x_1^-]][x_1, x_2]) - p_{21} q^2 (1 + q - q^{-1} - q^{-2})[x_2^-, [x_2^-, x_1^-]][x_1, x_2] + p_{21} q^2 (1 - q^{-1})(1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- = p_{21}(1 - q^{-3})(1 + q)((1 - q^{-2})(1 + q - q^{-1} - q^{-2}) + q^2(1 - q^{-1})(1 - q^{-2}))x_2^- = p_{21} q(1 + q)(1 - q^{-2})(1 - q^{-3})^2 x_2^-, \end{aligned}$$

$$\begin{aligned} \text{(A32)} \quad & [[x_2, [x_2, x_1]], x_1^-] = [x_2[x_2, x_1], x_1^-] - p_{22} p_{21} [[x_2, x_1]x_2, x_1^-] = p_{12} p_{11} [x_2, x_1^-][x_2, x_1] + \\ & x_2 [[x_2, x_1], x_1^-] - p_{22} p_{12} p_{21} [[x_2, x_1], x_1^-]x_2 - p_{22} p_{21} [x_2, x_1][x_2, x_1^-] = (1 - q^{-3})x_2^2 - q^{-2}(1 - q^{-3})x_2^2 = (1 - q^{-2})(1 - q^{-3})x_2^2, \end{aligned}$$

$$\begin{aligned} \text{(A33)} \quad & [[x_2, [x_2, x_1]], x_2^-] = [x_2[x_2, x_1], x_2^-] - p_{22} p_{21} [[x_2, x_1]x_2, x_2^-] = p_{22} p_{21} [x_2, x_2^-][x_2, x_1] + \\ & x_2 [[x_2, x_1], x_2^-] - p_{22} p_{22} p_{21} [[x_2, x_1], x_2^-]x_2 - p_{22} p_{21} [x_2, x_1][x_2, x_2^-] = p_{22} p_{21} (1 - g_2 f_2)[x_2, x_1] - p_{21} (1 - q^{-3})x_2 g_2 f_2 x_1 + p_{22}^2 p_{21}^2 (1 - q^{-3})g_2 f_2 x_1 x_2 - p_{22} p_{21} [x_2, x_1](1 - g_2 f_2) = -p_{21} q^2 (1 - q^{-3})g_2 f_2 [x_2, x_1] - p_{21} q g_2 f_2 [x_2, x_1] + p_{21} g_2 f_2 [x_2, x_1] = -p_{21} q(1 + q)(1 - q^{-2})g_2 f_2 [x_2, x_1], \end{aligned}$$

$$\begin{aligned} \text{(A34)} \quad & [[x_2, [x_2, x_1]], [x_1^-, x_2^-]] = [x_2, [[x_2, x_1], [x_1^-, x_2^-]]] + p_{12} p_{11} p_{22} p_{21} [[x_2, [x_1^-, x_2^-]], [x_2, x_1]] = \\ & (1 - q^{-3})x_2(1 - g_2 f_2 g_1 f_1) - p_{22} p_{21} p_{12} p_{22} (1 - q^{-3})(1 - g_1 f_1 g_2 f_2)x_2 + p_{12}(1 - q^3)x_1^- [x_2, x_1] - p_{22} p_{21} p_{12}^{-1} p_{11}^{-1} p_{22}^{-1} p_{12} (1 - q^3)[x_2, x_1]x_2^- = (1 - q^{-3})(1 - q^{-1})x_2 + p_{12}(1 - q^3)x_1^- [x_2, x_1] - q^{-3}(1 - q^3)((1 - q^{-3})x_2 + p_{12} q^3 x_1^- [x_2, x_1]) = (1 + q)(1 - q^{-2})(1 - q^{-3})x_2, \end{aligned}$$

$$\begin{aligned} \text{(A35)} \quad & [[x_2, [x_2, x_1]], [[x_1^-, x_2^-], x_2^-]] = [x_2, [[x_2, x_1], [[x_1^-, x_2^-], x_2^-]]] + p_{12} p_{22}^2 p_{11} p_{21}^2 \\ & [[x_2, [[x_1^-, x_2^-], x_2^-]], [x_2, x_1]] = (1 - q^{-3})(1 - q^{-2})(1 + q)x_2 x_2^- g_2 f_2 g_1 f_1 - \\ & p_{22} p_{21} p_{12} p_{22}^2 (1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- g_2 f_2 g_1 f_1 x_2 + p_{21} q^2 p_{12} (1 + q - q^2 - q^3)[x_1^-, x_2^-][x_2, x_1] - q^{-1} p_{22} p_{21} p_{12}^{-1} p_{11}^{-1} p_{22}^2 p_{21}^2 (1 + q - q^2 - q^3)[x_2, x_1][x_1^-, x_2^-] = \\ & (1 - q^{-3})(1 - q^{-2})(1 + q)(1 - g_2 f_2 + q x_2^- x_2)g_2 f_2 g_1 f_1 - (1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- g_2 f_2 g_1 f_1 x_2 + q^{-1}(1 + q - q^2 - q^3)[x_1^-, x_2^-][x_2, x_1] - q^{-2}(1 + q - q^2 - q^3)((1 - q^{-3})(1 - g_2 f_2 g_1 f_1) + p_{12} p_{11} p_{22} p_{21} [x_1^-, x_2^-][x_2, x_1]) = (1 + q)(1 - q^{-2})(1 - q^{-3})(1 - g_1 f_1 g_2^2 f_2^2), \end{aligned}$$

$$\begin{aligned} \text{(A36)} \quad & [[x_2, [x_2, x_1]], [[[x_1^-, x_2^-], x_2^-], x_2^-]] = [[[x_2, [x_2, x_1]], [[x_1^-, x_2^-], x_2^-]], x_2^-] + \\ & p_{12}^2 p_{11} p_{22}^4 p_{21}^2 [[[x_1^-, x_2^-], x_2^-], [x_2, [x_2, x_1]], x_2^-]] = (1 - q^{-3})(1 - q^{-2})(1 + q)(1 - g_2^2 f_2^2 g_1 f_1)x_2^- - p_{22}^2 p_{21} p_{21}^{-1} p_{22}^2 (1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- (1 - g_2^2 f_2^2 g_1 f_1) + \\ & q p_{21} (1 - q + q^{-1} - q^2)[[x_1^-, x_2^-], x_2^-]g_2 f_2 [x_2, x_1] - p_{12}^2 p_{11}^{-1} p_{22}^4 p_{21}^{-1} p_{22}^2 q p_{21} (1 - q + q^{-1} - q^2)g_2 f_2 [x_2, x_1][[x_1^-, x_2^-], x_2^-] = (1 - q^{-3})(1 - q^{-2})(1 + q)(1 - q)x_2^- g_2^2 f_2^2 g_1 f_1 + p_{21} q(1 - q + q^{-1} - q^2)[[x_1^-, x_2^-], x_2^-]g_2 f_2 [x_2, x_1] - q^{-2}(1 - q + q^{-1} - q^2)g_2 f_2 ((1 - q^{-3})(1 - q^{-2})(1 + q)x_2^- g_2^2 f_2^2 g_1 f_1 + p_{12} p_{11} p_{22}^2 p_{21}^2 [[x_1^-, x_2^-], x_2^-][x_2, x_1]) = (1 - q^{-3})(1 - q^{-2})(1 + q)(1 - q - 1 + q + q^{-1} - q^{-2})x_2^- g_2^2 f_2^2 g_1 f_1 = q^2(1 + q)(1 - q^{-2})(1 - q^{-3})^2 x_2^- g_1 f_1 g_2^2 f_2^2, \end{aligned}$$

$$(A37) \quad [[x_2, [x_2, x_1]], [x_1^-, x_2^-], [x_1^-, x_2^-], x_2^-]] = -p_{12}q^3(1+q)(1-q^{-2})(1-q^{-3})^2[x_1^-, x_2^-],$$

$$(A38) \quad \begin{aligned} &[[[x_1, x_2], x_2], x_2], x_1^-] = [[[x_1, x_2], x_2]x_2, x_1^-] - p_{12}p_{22}^2[x_2[[x_1, x_2], x_2], x_1^-] = \\ &p_{12}[[[x_1, x_2], x_2], x_1^-]x_2 + [[x_1, x_2], x_2][x_2, x_1^-] - p_{12}p_{22}^2p_{11}p_{12}^2[x_2, x_1^-][x_1, x_2], x_2] - \\ &p_{12}p_{22}^2x_2[[[x_1, x_2], x_2], x_1^-] = -p_{12}^3(1-q^{-3})(1-q^{-2})g_1f_1x_2^3 + p_{12}^3q^2(1-q^{-3})(1-q^{-2})p_{12}p_{21}g_1f_1x_2^3 = -p_{12}^3(1-q^{-1})(1-q^{-2})(1-q^{-3})g_1f_1x_2^3, \end{aligned}$$

$$(A39) \quad \begin{aligned} &[[[x_1, x_2], x_2], x_2], x_2^-] = [[[x_1, x_2], x_2]x_2, x_2^-] - p_{12}p_{22}^2[x_2[[x_1, x_2], x_2], x_2^-] = \\ &p_{22}[[[x_1, x_2], x_2], x_2^-]x_2 + [[x_1, x_2], x_2][x_2, x_2^-] - p_{12}p_{22}^4p_{21}[x_2, x_2^-][x_1, x_2], x_2] - \\ &p_{12}p_{22}^2x_2[[[x_1, x_2], x_2], x_2^-] = p_{22}(1+q-q^{-1}-q^{-2})[x_1, x_2]x_2 + [[x_1, x_2], x_2](1-g_2f_2) - \\ &q(1-g_2f_2)[x_1, x_2], x_2] - p_{12}q^2(1+q-q^{-1}-q^{-2})x_2[x_1, x_2] = \\ &q(1+q-q^{-1}-q^{-2})[[x_1, x_2], x_2] + (1-q)[x_1, x_2], x_2] - [[x_1, x_2], x_2]g_2f_2 + \\ &qg_2f_2[[x_1, x_2], x_2] = q^2(1-q^{-3})[[x_1, x_2], x_2], \end{aligned}$$

$$(A40) \quad \begin{aligned} &[[[x_1, x_2], x_2], x_2], [x_2^-, x_1^-]] = [[[x_1, x_2], x_2], [x_2, [x_2^-, x_1^-]]] + p_{22}p_{12}[[[x_1, x_2], x_2], \\ &[x_2^-, x_1^-]], x_2] = (1-q^{-3})[[x_1, x_2], x_2]x_1^-g_2f_2 - p_{12}p_{22}^2p_{21}p_{22}^2p_{11}p_{12}^2(1-q^{-3})x_1^-g_2f_2[[x_1, x_2], x_2] - \\ &p_{12}^2q^2(1-q^{-3})(1-q^{-2})(1+q)g_1f_1g_2f_2x_2^2 + p_{12}p_{22}^2p_{22}^2p_{12}^{-1}p_{12}^2q^2(1-q^{-3})(1-q^{-2})(1+q)x_2g_1f_1g_2f_2x_2 = \\ &(1-q^{-3})(-p_{12}^2(1-q^{-3})(1-q^{-2})g_1f_1x_2^2 + p_{11}p_{12}^2x_1^-[[x_1, x_2], x_2])g_2f_2 - p_{12}^2q^4(1-q^{-3})x_1^-g_2f_2[[x_1, x_2], x_2] = -p_{12}^2q^4(1-q^{-2})(1-q^{-3})^2g_1f_1g_2f_2x_2^2, \end{aligned}$$

$$(A41) \quad \begin{aligned} &[[[x_1, x_2], x_2], x_2], [x_2^-, [x_2^-, x_1^-]]] = [[[x_1, x_2], x_2], [x_2, [x_2^-, [x_2^-, x_1^-]]]] + p_{22}^2p_{12} \\ &[[[x_1, x_2], x_2], [x_2^-, [x_2^-, x_1^-]]], x_2] = (1+q-q^{-1}-q^{-2})[[x_1, x_2], x_2][x_2^-, x_1^-]g_2f_2 - \\ &p_{12}p_{22}^2p_{21}p_{22}^4p_{11}p_{12}^2(1+q-q^{-1}-q^{-2})[x_2^-, x_1^-]g_2f_2[[x_1, x_2], x_2] + p_{12}q^2(1-q^{-3})(1-q^{-2})(1+q)(1-g_1f_1g_2^2f_2^2)x_2 - \\ &p_{12}p_{22}^2p_{22}^2p_{12}^{-1}p_{12}q^2(1-q^{-3})(1-q^{-2})(1+q)x_2(1-g_1f_1g_2^2f_2^2) = (1+q-q^{-1}-q^{-2})(-p_{12}q(1-q^{-3})(1-q^{-2})(1+q)g_1f_1g_2f_2x_2 + \\ &p_{21}p_{22}^2p_{11}p_{12}^2[x_2^-, x_1^-][x_1, x_2], x_2])g_2f_2 - p_{12}q^3(1-q-q^{-1}-q^{-2})[x_2^-, x_1^-]g_2f_2[[x_1, x_2], x_2] - \\ &p_{12}q^2(1-q^{-3})(1-q^{-2})(1+q)g_1f_1g_2^2f_2^2x_2 + p_{12}(1-q^{-3})(1-q^{-2})(1+q)g_1f_1g_2^2f_2^2x_2 = -p_{12}q^3(1-q^{-3})(1-q^{-2})(1+q)(1+q-q^{-1}-q^{-2}+q^{-1}-1)g_1f_1g_2^2f_2^2x_2 = \\ &-p_{12}q^4(1+q)(1-q^{-2})(1-q^{-3})^2g_1f_1g_2^2f_2^2x_2, \end{aligned}$$

$$(A42) \quad \begin{aligned} &[[[x_1, x_2], x_2], x_2], [x_2^-, [x_2^-, [x_2^-, x_1^-]]]] = [[[x_1, x_2], x_2], [x_2, [x_2^-, [x_2^-, [x_2^-, x_1^-]]]]] + \\ &p_{22}^3p_{12}[[[x_1, x_2], x_2], [x_2^-, [x_2^-, [x_2^-, x_1^-]]], x_2] = q_2(1-q^{-3})[[x_1, x_2], x_2][x_2^-, [x_2^-, x_1^-]] \\ &g_2f_2 - p_{12}p_{22}^2p_{21}p_{22}^6p_{11}p_{12}^2q_2(1-q^{-3})[x_2^-, [x_2^-, x_1^-]]g_2f_2[[x_1, x_2], x_2] + p_{12}q^3p_{21}(1-q^{-3})(1-q^{-2})(1+q)(1-q^3)x_2^-x_2 - \\ &p_{12}p_{22}p_{22}^3p_{12}^{-1}p_{12}q^3p_{21}(1-q^{-3})(1-q^{-2})(1+q)(1-q^3)x_2^-x_2 = q^2(1-q^{-3})(1-q^{-3})(1-q^{-2})(1+q)(1-g_1f_1g_2^2f_2^2) + \\ &p_{21}p_{22}^4p_{11}p_{12}^2[x_2^-, [x_2^-, x_1^-]][x_1, x_2], x_2]g_2f_2 - q^4(1-q^{-3})[x_2^-, [x_2^-, x_1^-]]g_2f_2[[x_1, x_2], x_2] + \\ &(1-q^{-3})(1-q^{-2})(1+q)(1-q^3)x_2^-x_2 - q^{-1}(1-q^{-3})(1-q^{-2})(1+q)(1-q^3)(1-g_2f_2 + p_{22}x_2^-x_2) = \\ &q^2(1-q^{-3})^2(1-q^{-2})(1+q)(g_2f_2 - g_1f_1g_2^3f_2^3) + q^4(1-q^{-3})[x_2^-, [x_2^-, x_1^-]]g_2f_2[[x_1, x_2], x_2] - q^4(1-q^{-3})[x_2^-, [x_2^-, x_1^-]]g_2f_2[[x_1, x_2], x_2] - \\ &q^{-1}(1-q^{-3})(1-q^{-2})(1+q)(1-q^3)(1-g_2f_2) = q^2(1+q)(1-q^{-2})(1-q^{-3})^2(1-g_1f_1g_2^3f_2^3), \end{aligned}$$

$$(A43) \quad \begin{aligned} &[[x_2, [x_2, [x_2, x_1]]], x_2^-] = [x_2[x_2, [x_2, x_1]], x_2^-] - p_{22}^2p_{21}[[x_2, [x_2, x_1]]x_2, x_2^-] = \\ &p_{22}^2p_{21}[x_2, x_2^-][x_2, [x_2, x_1]] + x_2[x_2, [x_2, x_1]], x_2^-] - p_{22}^3p_{21}[[x_2, [x_2, x_1]], x_2^-]x_2 - \\ &p_{22}^2p_{21}[x_2, [x_2, x_1]][x_2, x_2^-] = p_{21}q^2(1-g_2f_2)[x_2, [x_2, x_1]] + p_{21}(1+q+q^{-1}-q^2)x_2g_2f_2[x_2, x_1] - q^3p_{21}^2(1+q+q^{-1}-q^2)g_2f_2[x_2, x_1]x_2 - p_{21}q^2[x_2, [x_2, x_1]](1- \end{aligned}$$

$$g_2 f_2 = -p_{21} q^2 g_2 f_2 [x_2, [x_2, x_1]] + p_{21} q^3 g_2 f_2 [x_2, [x_2, x_1]] + p_{21} q^2 (1 - q + q^{-1} - q^2) g_2 f_2 x_2 [x_2, x_1] - p_{21}^2 q^3 (1 - q + q^{-1} - q^2) g_2 f_2 [x_2, x_1] x_2 = p_{21} q (1 - q^{-3}) g_2 f_2 [x_2, [x_2, x_1]],$$

$$\begin{aligned} \text{(A44)} \quad & [[x_2, [x_2, [x_2, x_1]]], [[x_1^-, x_2^-], x_2^-] = [[x_2, [x_2, [x_2, x_1]]], [x_1^-, x_2^-], x_2^-] + p_{12}^3 p_{11} \\ & p_{22}^3 p_{21} [[x_1^-, x_2^-], [[x_2, [x_2, [x_2, x_1]]], x_2^-] = q^2 (1 - q^{-3})^2 (1 - q^{-2}) x_2^2 x_2^- - \\ & p_{22}^3 p_{21} p_{21}^{-1} p_2 2^{-1} q^2 (1 - q^{-3})^2 (1 - q^{-2}) x_2^- x_2^2 + p_{12}^2 q^3 p_{21} q (1 - q^3) [x_1^-, x_2^-] g_2 f_2 [x_2, [x_2, x_1]] - \\ & p_{12}^3 p_{11}^{-1} p_{22}^3 p_2 1^{-1} p_2 1^{-1} p_{22}^{-1} p_{12}^2 q^3 p_{21} q (1 - q^3) g_2 f_2 [x_2, [x_2, x_1]] [x_1, x_2^-] = q^2 (1 - \\ & q^{-3})^2 (1 - q^{-2}) ((1 + q) x_2 - q (1 + q) g_2 f_2 x_2 + q^2 x_2^- x_2^2) - q^4 (1 - q^{-3})^2 (1 - \\ & q^{-2}) x_2^- x_2^2 + p_{12} q (1 - q^3) [x_1^-, x_2^-] g_2 f_2 [x_2, [x_2, x_1]] - (1 - q^3) g_2 f_2 ((1 - q^{-3} (1 - \\ & q^{-2} (1 + q) x_2 + p_{12}^2 p_{22}^2 p_{11} p_{21} [x_1^-, x_2^-] [x_2, [x_2, x_1]] = q^2 (1 - q^{-3})^2 (1 - q^{-2}) (1 + \\ & q) x_2 - q^3 (1 - q^{-3})^2 (1 - q^{-2}) (1 + q) g_2 f_2 x_2 + q^3 (1 - q^{-3})^2 (1 - q^{-2}) (1 + \\ & q) g_2 f_2 x_2 = q^2 (1 + q) (1 - q^{-2}) (1 - q^{-3}) x_2, \end{aligned}$$

$$\begin{aligned} \text{(A45)} \quad & [[x_2, [x_2, [x_2, x_1]]], [[x_1^-, x_2^-], x_2^-] = [[x_2, [x_2, [x_2, x_1]]], [[x_1^-, x_2^-], x_2^-], x_2^-] + \\ & p_{12}^3 p_{11} p_{22}^2 p_{21}^2 [[x_1^-, x_2^-], x_2^-], [[x_2, [x_2, [x_2, x_1]]], x_2^-] = q^2 (1 - q^{-3})^2 (1 - q^{-2}) (1 + \\ & q) x_2 x_2^- - p_{22}^3 p_{21} p_{21}^{-1} p_2^2 q^2 (1 - q^{-3})^2 (1 - q^{-2}) (1 + q) x_2^- x_2 + p_{12} q^3 p_{21} q (1 - \\ & q^3) [[x_1^-, x_2^-], x_2^-] g_2 f_2 [x_2, [x_2, x_1]] - p_{21}^{-1} p_{22}^2 p_{12}^3 p_{11}^{-1} p_{22}^6 p_{21}^2 q (1 - q^3) g_2 f_2 ((1 - \\ & q^{-3}) (1 - q^{-2}) (1 + q) (1 - g_1 f_1 g_2^2 f_2^2) + p_{12}^2 p_{11} p_{22}^4 p_{21}^2 [[x_1^-, x_2^-], x_2^-] [x_2, [x_2, x_1]]) = \\ & q^2 (1 - q^{-3})^2 (1 - q^{-2} (1 + q) (1 - g_2 f_2) - q^{-1} (1 - q^3) (1 - q^{-3}) (1 - q^{-2}) (1 + \\ & q) (g_2 f_2 - g_1 f_1 g_2^3 f_2^3) = q^2 (1 + q) (1 - q^{-2}) (1 - q^{-3})^2 (1 - g_1 f_1 g_2^3 f_2^3), \end{aligned}$$

$$\begin{aligned} \text{(A46)} \quad & [[[x_1, x_2], [x_2, [x_2, x_1]]], x_2^-] = [[x_1, x_2], [[x_2, [x_2, x_1]], x_2^-] + p_{21} p_{22}^2 [[[x_1, x_2], x_2^-], \\ & [x_2, [x_2, x_1]]] = p_{21} (1 - q + q^{-1} - q^2) [x_1, x_2] g_2 f_2 [x_2, x_1] - p_{12}^2 p_{11} p_{22}^3 p_{21}^3 (1 - q + \\ & q^{-1} - q^2) g_2 f_2 [x_2, x_1] [x_1, x_2] + p_{21} q^2 (1 - q^{-3}) x_1 [x_2, [x_2, x_1]] - p_{12}^2 p_{11} p_{22}^2 p_{21} p_{22}^{-2} p_{21}^{-1} p_{21} q^2 (1 - \\ & q^{-3}) [x_2, [x_2, x_1]] x_1 = p_{21} q^{-1} (1 - q + q^{-1} - q^2) g_2 f_2 (q^3 p_{12} (1 - q^{-3}) x_2 x_1 - \\ & p_{12} q^3 [x_2, x_1]) [x_2, x_1] - p_{21} (1 - q + q^{-1} - q^2) g_2 f_2 [x_2, x_1] q^3 p_{12} (1 - q^{-3}) x_2 x_1 - \\ & p_{12} q^3 [x_2, x_1] + p_{21} q^2 (1 - q^{-3}) (p_{12} q [x_2, x_1]^2 + p_{12}^2 q^3 [x_2, [x_2, x_1]] x_1) - p_{12} q^2 (1 - \\ & q^{-3}) [x_2, [x_2, x_1]] x_1 = (1 - q^{-3}) [x_2, x_1]^2 + p_{12} q^3 (1 + q) (1 - q^{-2}) (1 - q^{-3})^2 g_2 f_2 x_2 [x_2, x_1] x_1 - \\ & p_{12} q^3 (1 + q) (1 - q^{-2}) (1 - q^{-3}) g_2 f_2 [x_2, [x_2, x_1]] x_1 + (1 + q)^2 (1 - q^{-2}) g_2 f_2 [x_2, x_1]^2, \end{aligned}$$

$$\begin{aligned} \text{(A47)} \quad & [[[x_1, x_2], [x_1, x_2], x_2], x_2^-] = [[x_1, x_2], [[[x_1, x_2], x_2], x_2^-] + p_{21} p_{22}^2 [[[x_1, x_2], x_2^-], \\ & [x_1, x_2], x_2] = (1 + q - q^{-1} - q^{-2}) [x_1, x_2]^2 - p_{11} p_{12}^2 p_{21} p_{22}^2 p_{21} p_{22} (1 + q - q^{-1} - \\ & q^{-2}) [x_1, x_2]^2 + p_{21} q^2 (1 - q^{-3}) x_1 [[x_1, x_2], x_2] - p_{11} p_{12}^2 p_{21} p_{22}^2 p_{21}^{-1} p_{22}^{-2} p_{21} q^2 (1 - \\ & q^{-3}) [[x_1, x_2], x_2] x_1 = -p_{12} q^2 (1 - q^{-3}) [[x_1, x_2], x_2] x_1 + p_{21} q^2 (1 - q^{-3}) (p_{12} q (1 - \\ & q^{-2}) [x_1, x_2]^2 + p_{12}^2 q^3 [[x_1, x_2], x_2] x_1) = (1 - q^{-2}) (1 - q^{-3}) [x_1, x_2]^2, \end{aligned}$$

$$\begin{aligned} \text{(A48)} \quad & [[[x_1, x_2], [x_1, x_2], x_2], x_1^-] = [[x_1, x_2], [[[x_1, x_2], x_2], x_1^-] + p_{11} p_{12} [[[x_1, x_2], x_1^-], \\ & [x_1, x_2], x_2] = -p_{12}^2 (1 - q^{-3}) (1 - q^{-2}) [x_1, x_2] g_1 f_1 x_2^2 + p_{11} p_{12}^2 p_{21} p_{22}^2 p_{11} p_{12} p_{12}^2 (1 - \\ & q^{-3}) (1 - q^{-2}) g_1 f_1 x_2^2 [x_1, x_2] - p_{12}^2 p_{11} (1 - q^{-3}) g_1 f_1 x_2 [x_1, x_2], x_2 + p_{11} p_{12}^2 p_{21} p_{22}^2 \\ & p_{11}^{-1} p_{12}^{-2} p_{12}^{-2} p_{11} (1 - q^{-3}) [[x_1, x_2], x_2] g_1 f_1 x_2 = -p_{12}^2 q^3 (1 - q^{-3}) (1 - q^{-2}) g_1 f_1 ([[[x_1, x_2], x_2], x_2] + \\ & p_{12} q (1 + q) x_2 [[x_1, x_2], x_2] + p_{12}^2 q^2 x_2^2 [x_1, x_2]) + p_{12}^4 q^5 (1 - q^{-3}) (1 - q^{-2}) g_1 f_1 x_2^2 [x_1, x_2] - \\ & p_{12}^2 q^3 (1 - q^{-3}) g_1 f_1 x_2 [[x_1, x_2], x_2] + p_{12} q^2 (1 - q^{-3}) g_1 f_1 ([[[x_1, x_2], x_2], x_2] + \\ & p_{12} q^2 x_2 [[x_1, x_2], x_2]) = -p_{12}^2 q^3 (1 - q^{-3}) g_1 f_1 [[x_1, x_2], x_2], x_2 - p_{12}^3 q^4 (1 + \\ & q) (1 - q^{-2}) (1 - q^{-3}) g_1 f_1 x_2 [[x_1, x_2], x_2], \end{aligned}$$

$$\begin{aligned} \text{(A49)} \quad & [[[x_1, x_2], [x_1, x_2], x_2], [x_2^-, x_1^-]] = [[x_1, x_2], [[[x_1, x_2], x_2^-], [x_2^-, x_1^-]]] + p_{21} p_{22} p_{11} p_{12} \\ & [[[x_1, x_2], [x_2^-, x_1^-]], [[x_1, x_2], x_2]] = -p_{12} q (1 - q^{-3}) (1 - q^{-2}) (1 + q) [x_1, x_2] g_1 f_1 g_2 f_2 x_2 + \end{aligned}$$

$$\begin{aligned}
& p_{11}p_{12}^2p_{21}p_{22}^2p_{21}p_{22}p_{11}p_{12}p_{12}q(1-q^{-3})(1-q^{-2})(1+q)g_1f_1g_2f_2x_2[x_1, x_2] + \\
& q(1-q^{-3})(1-g_1f_1g_2f_2)[[x_1, x_2], x_2] - p_{11}p_{12}^2p_{21}p_{22}^2p_{21}^{-1}p_{22}^{-1}p_{11}^{-1}p_{12}^{-2}q(1-q^{-3}) \\
& [[x_1, x_2], x_2](1-g_1f_1g_2f_2) = -p_{12}q^3(1-q^{-3})(1-q^{-2})(1+q)g_1f_1g_2f_2([x_1, x_2], x_2) - \\
& p_{12}qx_2[x_1, x_2] + p_{12}q^4(1-q^{-3})(1-q^{-2})(1+q)g_1f_1g_2f_2x_2[x_1, x_2] + q(1-q^{-3}) \\
& (1-g_1f_1g_2f_2)[[x_1, x_2], x_2] - q(1-q^{-3})[[x_1, x_2], x_2] + q(1-q^{-3})qg_1f_1g_2f_2[[x_1, x_2], x_2] = \\
& -p_{12}q^4(1-q^{-3})^2g_1f_1g_2f_2[[x_1, x_2], x_2],
\end{aligned}$$

$$\begin{aligned}
\text{(A50)} \quad & [[[x_1, x_2], [x_1, x_2], x_2], [x_2^-, [x_2^-, x_1^-]]] = [[x_1, x_2], [[[x_1, x_2], x_2], [x_2^-, [x_2^-, x_1^-]]]] + \\
& p_{21}^2p_{22}^4p_{11}p_{12}^2[[[x_1, x_2], [x_2^-, [x_2^-, x_1^-]]], [[x_1, x_2], x_2]] = (1-q^{-3})(1-q^{-2})(1+q) \\
& [x_1, x_2](1-g_1f_1g_2^2f_2^2) - p_{11}p_{12}^2p_{21}p_{22}^2p_{21}^2p_{22}^2p_{11}p_{12}(1-q^{-3})(1-q^{-2})(1+q) \\
& (1-g_1f_1g_2^2f_2^2)[x_1, x_2] + qp_{21}(1-q^{-3})(1-q^2)(1+q)x_2^-[[x_1, x_2], x_2] - \\
& p_{11}p_{12}^2p_{21}p_{22}^2p_{21}^{-2}p_{22}^{-4}p_{11}^{-1}p_{12}^{-2}qp_{21}(1-q^{-3})(1-q^2)(1+q)((1+q-q^{-1}-q^{-2}) \\
& [x_1, x_2] + p_{21}q^2x_2^-[[x_1, x_2], x_2]) = (1-q^{-3})(1-q^{-2})(1+q)(1-q)[x_1, x_2] + \\
& p_{21}q(1-q^{-3})(1-q^2)(1+q)x_2^-[[x_1, x_2], x_2] - q^{-1}(1-q^{-3})(1-q^2)(1+q) \\
& (1+q-q^{-1}-q^{-2})[x_1, x_2] + p_{21}q(1-q^{-3})(1-q^2)(1+q)x_2^-[[x_1, x_2], x_2] = \\
& q^2(1+q)(1-q^{-2})(1-q^{-3})^2[x_1, x_2].
\end{aligned}$$

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